Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 14, 2004*
Revised: *January 21, 2004*
Accepted: *January 21, 2004*

$\mathcal{N} = 2$ supersymmetric $\mathrm{U}(1)$ gauge theory in noncommutative harmonic superspace

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Abstract: We study $\mathcal{N} = 2$ supersymmetric U(1) gauge theory in the noncommutative harmonic superspace with nonanticommutative fermionic coordinates. We examine the gauge transformation which preserves the Wess-Zumino gauge by harmonic expansions of component fields. The gauge transformation is shown to depend on the deformation parameters and the anti-holomorphic scalar field. We compute the action explicitly up to the third order in component fields and discuss the field redefinitions so that the component fields transform canonically.

Keywords: Gauge Symmetry, Extended Supersymmetry, Superspaces, Non-Commutative Geometry.

Contents

1. Introduction

Field theories on noncommutative superspace [1] have been taken much attentions recently. In superstring theory, they correspond to the field theories on the D-branes in the graviphoton background [2]. The Ramond-Ramond fields induces the non(anti)commutativity for the worldsheet fermionic fields in the hybrid formalism of superstrings [2, 3, 4]. When we consider the superstring theory compactified on a Calabi-Yau threefold, the field theory on the D-branes is described by $\mathcal{N} = 1$ super Yang-Mills theory on the noncommutative superspace. This theory has been also constructed in terms of chiral superfields [5], which has been shown to have $\mathcal{N} = \frac{1}{2}$ supersymmetry.

From the field theoretical viewpoint, noncommutative superspace provides various interesting problems. Compare to the field theories on noncommutative spacetime, only a finite number of the deformation terms are necessary in the case of $\mathcal{N} = 1$ noncommutative superspace. This property makes the deformed lagrangian simple, so that the perturbative [6] and nonperturbative structure [7] of field theories can be studied explicitly.

It is an interesting problem to study the supersymmetric gauge theory on the noncommutative extended superspace [8, 9, 10]. In a previous paper [11], we investigated the $\mathcal{N} = 2$ super Yang-Mills theory on the $\mathcal{N} = 1$ noncommutative superspace by adding adjoint chiral superfields to $\mathcal{N} = 1$ super Yang-Mills theory. Since the deformation of $\mathcal{N} = 1$ superspace breaks the $\mathcal{R}$-symmetry explicitly, the theory has only $\mathcal{N} = 1/2$ supersymmetry. Noncommutative $\mathcal{N} = 2$ superspace provides another approach to study the deformation of

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$\mathcal{N}=2$ supersymmetric gauge theories. $\mathcal{N}=2$ rigid superspace is convenient to construct the on-shell lagrangian of $\mathcal{N}=2$ supersymmetric Yang-Mills theory [12]. The lagrangian is defined by chiral superfields, which obeys certain constrains. These constraints connects the chiral part and anti-chiral parts. One may consider the noncommutative deformation of rigid superspace. It turns out that the constraint equations are highly nontrivial to solve.

Harmonic superspace [13] is known to provide a good off-shell formulation of quantum field theory with extended supersymmetry. This superspace is obtained by adding the harmonic coordinates of S^2 to the rigid $\mathcal{N}=2$ superspace. In this formalism, analytic superfields play an important role instead of $\mathcal{N}=2$ chiral superfields. These superfields are unconstrained but includes infinitely many fields in their components arising from the harmonic expansions.

In refs. [10, 9], the noncommutative harmonic superspace has been considered. Ferrara and Sokatchev [10] studied $\mathcal{N}=2$ U(1) gauge theories based on the singlet deformation whose Poisson structure includes only the supercovariant derivatives (singlet D deformation). This Poisson structure preserves $\mathcal{N}=2$ supersymmetry and analyticity but does not preserve chiral structure of the theory. The lagrangian depends on certain function of anti-holomorphic scalar field. Ivanov *et al.* have studied the deformation of $\mathcal{N}=2$ harmonic superspace [9], whose Poisson structure is characterized by the bi-differential operator with supercharges called the Q deformation in [10]. The analytic superfields are also well-defined in this deformation. But the only the chiral part of $\mathcal{N}=2$ supersymmetry are realized.

In this paper, we study the Q deformation of $\mathcal{N}=2$ harmonic superspace and $\mathcal{N}=2$ supersymmetric U(1) gauge theory in this noncommutative superspace. We will consider the most general deformation in the harmonic superspace. We examine the gauge transformation of the theory in detail in terms of component fields. The action of $\mathcal{N}=2$ U(1) gauge theory, as observed in the case of the singlet D deformation [10], is shown to contain arbitrary power of anti-holomorphic scalar fields. This means also the action contains infinite number of deformation parameters. In this paper, we will examine the action up to the third order contributions in component fields.

This paper is organized as follows: In section 2, we review $\mathcal{N}=2$ harmonic superspace and study its deformation and the Poisson structure. In section 3, we discuss the lagrangian of the $\mathcal{N}=2$ supersymmetric U(1) gauge theory on the $\mathcal{N}=2$ harmonic superspace. We will calculate the lagrangian up to the third order in component fields explicitly. In section 4, we will study the gauge transformation in the deformed harmonic superspace and discuss the field redefinitions. In the appendix A, we summarize the Poisson structure in the analytic basis. In the appendix B, we will calculate the action of the third order in component fields explicitly.

2. Noncommutative harmonic superspace

In this section, we introduce noncommutative harmonic superspace and study the Poisson structure. We begin with reviewing the $\mathcal{N}=2$ harmonic superspace which has been explained in detail in [13]. Let $(x^\mu, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha} i})$ be the coordinates of the $\mathcal{N}=2$ rigid superspace, where $\mu=0,1,2,3$ are indices of spacetime coordinates, $\alpha, \dot{\alpha}=1,2$ spinor indices and

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$i = 1, 2$ labels the SU(2) doublet. The signature of spacetime is euclidean. Concerning spinor indices, we will follow the notation in ref. [14]. Namely spinors with upper and lower indices are related through the ε-tensor with $\varepsilon^{12} = -\varepsilon_{12} = 1$. On the other hand, raising and lowering SU(2) indices are done with the help of antisymmetric tensor ϵ_{ij} with $\epsilon^{12} = -\epsilon_{12} = -1$, $\theta^i = \epsilon^{ij}\theta_j$ and $\theta_i = \epsilon_{ij}\theta^j$.

The supersymmetry generators Q^i_α, $\bar{Q}_{\dot{\alpha}i}$ and the supercovariant derivatives D^i_α, $\bar{D}_{\dot{\alpha}i}$ are defined by

$$\begin{aligned} Q^i_\alpha &= \frac{\partial}{\partial\theta^\alpha_i} - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}\frac{\partial}{\partial x^\mu}\,, \qquad & \bar{Q}_{\dot{\alpha}i} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} + i\theta^\alpha_i(\sigma^\mu)_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu} \\ D^i_\alpha &= \frac{\partial}{\partial\theta^\alpha_i} + i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}\frac{\partial}{\partial x^\mu}\,, \qquad & \bar{D}_{\dot{\alpha}i} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} - i\theta^\alpha_i(\sigma^\mu)_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}\,. \end{aligned} \tag{2.1}$$

They satisfy the anticommutation relations

$$\begin{aligned} \{D^i_\alpha, D^j_\beta\} &= \{\bar{D}_{\dot{\alpha}i}, \bar{D}_{\dot{\beta}j}\} = 0\,, \qquad \{D^i_\alpha, \bar{D}_{\dot{\beta}j}\} = -2i\delta^i_j(\sigma^\mu)_{\alpha\dot{\beta}}\frac{\partial}{\partial x^\mu}\,, \\ \{Q^i_\alpha, Q^j_\beta\} &= \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0\,, \qquad \{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2i\delta^i_j(\sigma^\mu)_{\alpha\dot{\beta}}\frac{\partial}{\partial x^\mu}\,, \\ \{D^i_\alpha, Q^j_\beta\} &= \{D^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = \{\bar{D}_{\dot{\alpha}i}, Q^j_\beta\} = \{\bar{D}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0\,. \end{aligned} \tag{2.2}$$

We introduce the left-handed chiral basis $(x^\mu_L, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha}i})$ in superspace, where

$$x^\mu_L = x^\mu + i\theta_i\sigma^\mu\bar{\theta}^i\,. \tag{2.3}$$

In this basis, supercharges and supercovariant derivatives take the form

$$\begin{aligned} Q^i_\alpha &= \frac{\partial}{\partial\theta^\alpha_i}\,, \qquad & \bar{Q}_{i\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{i\dot{\alpha}}} + 2i\theta^\alpha_i\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu_L} \\ D^i_\alpha &= \frac{\partial}{\partial\theta^\alpha_i} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}\frac{\partial}{\partial x^\mu_L} \qquad & \bar{D}_{\dot{\alpha}i} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}}\,. \end{aligned} \tag{2.4}$$

In the $\mathcal{N} = 2$ rigid superspace, we introduce nonanticommutative deformation:

$$\{\theta^\alpha_i, \theta^\beta_j\}_* = C^{\alpha\beta}_{ij}\,, \tag{2.5}$$

Here the anticommutation relation is calculated by using the $*$-product, which is defined by

$$f * g(\theta) = f(\theta)\exp\left\{-\frac{1}{2}\overleftarrow{\frac{\partial}{\partial\theta^\alpha_i}}C^{\alpha\beta}_{ij}\overrightarrow{\frac{\partial}{\partial\theta^\beta_j}}\right\}g(\theta) \tag{2.6}$$

for functions f and g of θ. We assume that x^μ_L and $\bar{\theta}_{\dot{\alpha}i}$ (anti-)commute with other coordinates

$$\begin{aligned} [x^\mu_L, x^\nu_L]_* &= [x^\mu_L, \theta^\alpha_i]_* = [x^\mu_L, \bar{\theta}^{\dot{\alpha}i}]_* = 0\,, \\ \{\bar{\theta}^{\dot{\alpha}i}, \bar{\theta}^{\dot{\beta}j}\}_* &= \{\bar{\theta}^{\dot{\alpha}i}, \theta^\alpha_j\}_* = 0\,, \end{aligned} \tag{2.7}$$

which is a generalization of $\mathcal{N}=1$ noncommutative superspace [2, 5]. $C_{ij}^{\alpha\beta}$ is symmetric under the exchange of pairs of indices (αi),(βj): $C_{ij}^{\alpha\beta}=C_{ji}^{\beta\alpha}$. We decompose the nonanticommutative parameter $C_{ij}^{\alpha\beta}$ into symmetric and antisymmetric parts with respect to SU(2) indices, such as

$$C_{ij}^{\alpha\beta}=C_{(ij)}^{\alpha\beta}+\frac{1}{4}\epsilon_{ij}\varepsilon^{\alpha\beta}C_s\,. \tag{2.8}$$

Here C_s corresponds to the singlet deformation parameter [10]. $C_{(ij)}^{\alpha\beta}$ is symmetric with respect to i and j, and also α and β. Since x_L^μ are commuting variables, spacetime coordinates x^μ have nontrivial (anti-)commutations:

$$\begin{aligned} [x^\mu,x^\nu]_* &= C_{(ij)}^{\mu\nu}\bar\theta^i\bar\theta^j+\frac{1}{2}C_s\epsilon_{ij}\bar\theta^i\bar\sigma^{\mu\nu}\bar\theta^j \\ [x^\mu,\theta_i^\alpha]_* &= iC_{ij}^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}\bar\theta^{\dot\beta j} \end{aligned} \tag{2.9}$$

where

$$C_{(ij)}^{\mu\nu}=C_{(ij)}^{\alpha\beta}(\sigma^{\mu\nu})_\alpha{}^\gamma\varepsilon_{\beta\gamma} \tag{2.10}$$

and $\sigma^{\mu\nu}=\frac{1}{4}(\sigma^\mu\bar\sigma^\nu-\sigma^\nu\bar\sigma^\mu)$, $\bar\sigma^{\mu\nu}=\frac{1}{4}(\bar\sigma^\mu\sigma^\nu-\bar\sigma^\nu\sigma^\mu)$.

We next discuss $\mathcal{N}=2$ harmonic superspace. It is formulated by introducing harmonic variables $u_i^\pm$ which form a 2×2 SU(2) matrix. The harmonic variables satisfy the condition

$$u^{+i}u_i^-=1\,,\qquad \overline{u^{+i}}=u_i^-\,. \tag{2.11}$$

The coordinates of the harmonic superspace are $(x^\mu,\theta_i^\alpha,\bar\theta_{\dot\alpha i},u_i^\pm)$. Using $u_i^\pm$, SU(2) indices can be projected into two parts which have ±1 U(1) charges,

$$D_\alpha^\pm=u_i^\pm D_\alpha^i\,,\qquad \bar D_\alpha^\pm=u_i^\pm\bar D_\alpha^i\,. \tag{2.12}$$

D_α^i is solved by $D_\alpha^\pm$ such as $D_{\alpha i}=u_i^+D_\alpha^- - u_i^-D_\alpha^+$ with the help of the completeness condition $u_i^+u_j^- - u_j^+u_i^-=\epsilon_{ij}$. In the harmonic superspace, an important notion is an analytic superfield rather than chiral superfield satisfying $\bar D_{\dot\alpha}^i\Phi=0$. An analytic superfield $\Phi(x,\theta,\bar\theta,u)$ is defined by

$$D_\alpha^+\Phi=\bar D_{\dot\alpha}^+\Phi=0\,, \tag{2.13}$$

which can be conveniently written in terms of analytic coordinates:

$$x_A^\mu = x^\mu-i(\theta^i\sigma^\mu\bar\theta^j+\theta^j\sigma^\mu\bar\theta^i)u_i^+u_j^- = x^\mu-i(\theta^+\sigma^\mu\bar\theta^-+\theta^-\sigma^\mu\bar\theta^+)\,, \tag{2.14}$$

$$\theta_\alpha^\pm = u_i^\pm\theta_\alpha^i\,,\qquad \bar\theta_{\dot\alpha}^\pm=u_i^\pm\bar\theta_{\dot\alpha}^i\,. \tag{2.15}$$

The supercharges and supercovariant derivatives take the form

$$\begin{aligned} Q_\alpha^+ &= \frac{\partial}{\partial\theta^{-\alpha}}-2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{+\dot\alpha}\frac{\partial}{\partial x_A^\mu}\,, & Q_\alpha^- &= -\frac{\partial}{\partial\theta^{+\alpha}} \\ \bar Q_{\dot\alpha}^+ &= \frac{\partial}{\partial\bar\theta^{-\dot\alpha}}+2i\theta^{+\alpha}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x_A^\mu}\,, & \bar Q_{\dot\alpha}^- &= -\frac{\partial}{\partial\bar\theta^{+\dot\alpha}} \\ D_\alpha^+ &= \frac{\partial}{\partial\theta^{-\alpha}}\,, & D_\alpha^- &= -\frac{\partial}{\partial\theta^{+\alpha}}+2i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{-\dot\alpha}\frac{\partial}{\partial x_A^\mu}\,, \\ \bar D_{\dot\alpha}^+ &= \frac{\partial}{\partial\bar\theta^{-\dot\alpha}}\,, & \bar D_{\dot\alpha}^- &= -\frac{\partial}{\partial\bar\theta^{+\dot\alpha}}-2i\theta^{-\alpha}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x_A^\mu}\,. \end{aligned} \tag{2.16}$$

Their nontrivial anticommutation relations are

$$
\begin{aligned}
\{Q^+_\alpha, \bar{Q}^-_{\dot\alpha}\} &= 2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu_A}\,, & \{Q^-_\alpha, \bar{Q}^+_{\dot\alpha}\} &= -2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu_A}\,, \\
\{D^+_\alpha, \bar{D}^-_{\dot\alpha}\} &= -2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu_A}\,, & \{\bar{D}^+_{\dot\alpha}, D^-_\alpha\} &= 2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu_A}\,.
\end{aligned} \tag{2.17}
$$

An analytic superfield Φ is functions of $(x^\mu_A, \theta^+, \bar\theta^+, u)$,

$$
\Phi = \Phi(x^\mu_A, \theta^+, \bar\theta^+, u)\,. \tag{2.18}
$$

Expanding in θ, Φ takes the form

$$
\begin{aligned}
\Phi =\ & \phi(x_A, u) + \theta^+\psi(x_A, u) + \bar\theta^+\bar\chi(x_A, u) + (\theta^+)^2 M(x_A, u) + (\bar\theta^+)^2 N(x_A, u) + \\
& +\theta^+\sigma^\mu\bar\theta^+ A_\mu(x_A, u) + (\bar\theta^+)^2\theta^+\lambda(x_A, u) + (\theta^+)^2\bar\theta^+\bar\kappa(x_A, u) + (\theta^+)^2(\bar\theta^+)^2 D(x_A, u)\,.
\end{aligned} \tag{2.19}
$$

and each θ-components are expanded in the harmonic variables, e.g.,

$$
\phi(x_A, u) = \sum_{n=0}^{\infty} \phi^{(i_1\cdots i_{q+n}j_1\cdots j_n)}(x_A) u^+_{(i_1}\cdots u^+_{i_{q+n}} u^-_{j_1}\cdots u^-_{j_n)} \tag{2.20}
$$

for the field with U(1) charge $q \geq 0$. Similar formula holds for $q < 0$. Here we define $A_{(i_1\cdots i_n)}$ as the symmetrized sum over the permutation of indices $i_1, \cdots i_n$:

$$
A_{(i_1\cdots i_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} A_{i_{\sigma(1)}\cdots i_{\sigma(n)}} \tag{2.21}
$$

and S_n denotes the permutation group of degree n. We define the harmonic derivatives $D^{\pm\pm}$ and D^0 by

$$
D^{\pm\pm} = u^{+i}\frac{\partial}{\partial u^{-i}}\,, \qquad D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\,. \tag{2.22}
$$

They form an SU(2) algebra. In the analytic basis, they are expressed as

$$
\begin{aligned}
D^{\pm\pm} &= \partial^{++} - 2i\theta^\pm\sigma^\mu\bar\theta^\pm\frac{\partial}{\partial x^\mu_A} + \theta^{\pm\alpha}\frac{\partial}{\partial\theta^{\mp\alpha}} + \bar\theta^{\pm\dot\alpha}\frac{\partial}{\partial\bar\theta^{\mp\dot\alpha}}\,, \\
D^0 &= \partial^0 + \theta^{+\alpha}\frac{\partial}{\partial\theta^{+\alpha}} - \theta^{-\alpha}\frac{\partial}{\partial\theta^{-\alpha}} + \bar\theta^{+\dot\alpha}\frac{\partial}{\partial\bar\theta^{+\dot\alpha}} - \bar\theta^{-\dot\alpha}\frac{\partial}{\partial\bar\theta^{-\dot\alpha}}\,,
\end{aligned} \tag{2.23}
$$

where $\partial^{\pm\pm}$ and ∂^0 denote the harmonic derivatives with fixed x_A, $\theta^\pm$ and $\bar\theta^\pm$.

When we introduce the deformation (2.5) into the harmonic superspace, we note that the derivative in the $*$-product (2.6) is taken with the fixed chiral coordinates x_L. Therefore it is convenient to write the $*$-product of the form

$$
f * g(\theta) = f(\theta)\exp(P)g(\theta)\,, \qquad P = -\frac{1}{2}\overleftarrow{Q^i_\alpha} C^{\alpha\beta}_{ij}\overrightarrow{Q^j_\beta}\,. \tag{2.24}
$$

In the chiral basis, this formula leads to (2.6) due to (2.4). We call P the Poisson structure. In the analytic basis, one can compute the $*$-product by using $Q^i_\alpha = u^{+i}Q^-_\alpha - u^{-i}Q^+_\alpha$. For example we have

$$
\begin{aligned}
\{\theta^\eta, \theta^{\eta'}\}_* &= C^{\eta\eta'\alpha\beta}\,, \qquad (\eta, \eta' = \pm) \\
[x^\mu_A, x^\nu_A]_* &= 4C^{--\mu\nu}(\bar\theta^+)^2 \\
[x^\mu_A, \theta^\pm_\alpha]_* &= -2iC^{-\eta\beta\alpha}(\sigma^\mu\bar\theta^+)_\beta
\end{aligned} \tag{2.25}
$$

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where $C^{\eta\eta'\mu\nu} = u^{\eta i}u^{\eta' j}C^{\mu\nu}_{ij}$. For superfields A and B, the $*$-product takes the form

$$A * B = AB + APB + \frac{1}{2}AP^2B + \frac{1}{6}AP^3B + \frac{1}{24}AP^4B\,, \qquad P^5 = 0\,. \tag{2.26}$$

We note that the Poisson structure P commutes with the supercovariant derivatives. Therefore the analytic structure is preserved when we introduce noncommutativity in harmonic superspace. Concerning supersymmetry, only Q^i_α are conserved. Field theory on this noncommutative superspace has chiral $\mathcal{N}=1$ (or $\mathcal{N}=(1,0)$ in the sense of [9]) supersymmetry.

We now investigate the $*$-product for analytic superfields in detail for later use. There are two ways of calculating this $*$-product. One way is to use the formula (2.24) directly for analytic superfields. This approach is helpful to keep the Grassmann analytic structure of the theory manifestly. Since D^+_α and $\bar{D}^+_{\dot\alpha}$ commute with the Poisson structure P, the product $A * B$ is also analytic if A and B are analytic. Another way is to expand the analytic superfields in terms of chiral basis and to apply the formula (2.6). The latter approach is relatively easier to perform and be useful for the calculation of the action, which will be explained in detail in the next section.

Firstly we study the $*$-product in the analytic basis. Using P in (2.24), $\exp P$ becomes

$$\begin{aligned}
\exp P = 1 &- \frac{1}{2}\overleftarrow{Q^\alpha_i}C^{\alpha\beta}_{ij}\overrightarrow{Q^j_\beta} + \frac{1}{8}\overleftarrow{Q^{\alpha_1}_{i_1}}\overleftarrow{Q^{\alpha_2}_{i_2}}C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}\overrightarrow{Q^{j_2}_{\beta_2}}\overrightarrow{Q^{j_1}_{\beta_1}} - \\
&-\frac{1}{48}\overleftarrow{Q^{\alpha_1}_{i_1}}\overleftarrow{Q^{\alpha_2}_{i_2}}\overleftarrow{Q^{\alpha_3}_{i_3}}C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}C^{\alpha_3\beta_3}_{i_3j_3}\overrightarrow{Q^{j_3}_{\beta_3}}\overrightarrow{Q^{j_2}_{\beta_2}}\overrightarrow{Q^{j_1}_{\beta_1}} + \\
&+\frac{1}{384}\overleftarrow{Q^{\alpha_1}_{i_1}}\overleftarrow{Q^{\alpha_2}_{i_2}}\overleftarrow{Q^{\alpha_3}_{i_3}}\overleftarrow{Q^{\alpha_4}_{i_4}}C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}C^{\alpha_3\beta_3}_{i_3j_3}C^{\alpha_4\beta_4}_{i_4j_4}\overrightarrow{Q^{j_4}_{\beta_4}}\overrightarrow{Q^{j_3}_{\beta_3}}\overrightarrow{Q^{j_2}_{\beta_2}}\overrightarrow{Q^{j_1}_{\beta_1}}\,.
\end{aligned} \tag{2.27}$$

By computing the action of supercharges on the analytic superfield, one can compute the $*$-product. For the analytic superfield Φ in (2.19), we obtain

$$\begin{aligned}
Q^i_\alpha\Phi = &-u^{+i}\psi_\alpha - 2u^{+i}\theta^+_\alpha M + \bar\theta^{+\dot\alpha}\left\{-u^{+i}\sigma^\mu_{\alpha\dot\alpha}A_\mu + 2iu^{-i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi\right\} - \\
&-(\bar\theta^+)^2\left\{u^{+i}\lambda_\alpha + iu^{-i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\chi^{\dot\alpha}\right\} + (\theta^+\sigma^\mu\bar\theta^+)\left\{u^{+i}(\sigma^\mu\bar\kappa)_\alpha - iu^{-i}(\partial_\nu\psi\sigma^\mu\bar\sigma^\nu)^\gamma\varepsilon_{\alpha\gamma}\right\} + \\
&+(\bar\theta^+)^2\theta^{+\beta}\left\{-u^{+i}2\varepsilon_{\alpha\beta}D - iu^{-i}(\sigma^\nu\bar\sigma^\mu)_\beta{}^\gamma\varepsilon_{\alpha\gamma}\partial_\mu A_\nu\right\} + 2iu^{-i}(\theta^+)^2\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{+\dot\alpha}\partial_\mu M - \\
&-u^{-i}i(\theta^+)^2(\bar\theta^+)^2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\kappa^{\dot\alpha}\,.
\end{aligned} \tag{2.28}$$

Since $Q^i_\alpha\Phi$ is also of the form (2.19), we can apply the above formula to compute $Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}\Phi$ etc., which is given in the appendix A. The $*$-product $A * B$ can be expressed as

$$\begin{aligned}
A * B = AB &- (-1)^{|A|}\frac{1}{2}Q^i_\alpha A C^{\alpha\beta}_{ij}Q^j_\beta B - \frac{1}{8}(Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}A)C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}(Q^{j_2}_{\beta_2}Q^{j_1}_{\beta_1})B + \\
&+\frac{1}{48}(-1)^{|A|}(Q^{i_3}_{\alpha_3}Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}A)C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}C^{\alpha_3\beta_3}_{i_3j_3}(Q^{j_3}_{\beta_3}Q^{j_2}_{\beta_2}Q^{j_1}_{\beta_1}B) + \\
&+\frac{1}{384}(Q^{i_4}_{\alpha_4}Q^{i_3}_{\alpha_3}Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}A)C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}C^{\alpha_3\beta_3}_{i_3j_3}C^{\alpha_4\beta_4}_{i_4j_4}(Q^{j_4}_{\beta_4}Q^{j_3}_{\beta_3}Q^{j_2}_{\beta_2}Q^{j_1}_{\beta_1}B)\,.
\end{aligned} \tag{2.29}$$

where $|A| = 0(1)$ if A is Grassmann even (odd).

We next compute the $*$-product in the chiral basis. The analytic coordinates x^μ_A is related to the chiral coordinates x^μ_L by

$$x^\mu_A = x^\mu_L - 2i\theta^-\sigma^\mu\bar\theta^+\,. \tag{2.30}$$

A function $f(x_A)$ of x_A is expanded in the chiral basis

$$f(x_A) = f(x_L) - 2i\theta^-\sigma^\mu\bar{\theta}^+\partial_\mu f(x_L) + (\theta^-)^2(\bar{\theta}^+)^2\partial^\mu\partial_\mu f(x_L)\,. \tag{2.31}$$

Using this formula, we can express an analytic superfield Φ in (2.19) in the chiral basis:

$$\begin{aligned}\Phi =\; & \phi(x_L) + \theta^+\psi(x_L) + \bar{\theta}^+\bar{\chi}(x_L) + \\ & + (\theta^+)^2 M(x_L) + (\bar{\theta}^+)^2 N(x_L) + \theta^+\sigma^\mu\bar{\theta}^+ A_\mu(x_L) - 2i\theta^-\sigma^\mu\bar{\theta}^+\partial_\mu\phi(x_L) - \\ & - 2i(\theta^-\sigma^\mu\bar{\theta}^+)\theta^+\partial_\mu\psi(x_L) + i(\bar{\theta}^+)^2\theta^-\sigma^\mu\partial_\mu\bar{\chi}(x_L) + (\bar{\theta}^+)^2\theta^+\lambda + (\theta^+)^2\bar{\theta}^+\bar{\kappa}(x_L) + \\ & + (\theta^-)^2(\bar{\theta}^+)^2\partial^\mu\partial_\mu\phi(x_L) - 2i(\theta^+)^2(\theta^-\sigma^\mu\bar{\theta}^+)\partial_\mu M(x_L) - i(\bar{\theta}^+)^2\theta^+\sigma^\mu\bar{\sigma}^\nu\theta^-\partial_\nu A_\mu(x_L) + \\ & + (\theta^+)^2(\bar{\theta}^+)^2 D(x_L) + (\theta^-)^2(\bar{\theta}^+)^2\theta^+\partial^\mu\partial_\mu\psi(x_L) + i(\theta^+)^2(\bar{\theta}^+)^2\theta^-\sigma^\mu\partial_\mu\bar{\kappa}(x_L) + \\ & + (\theta^+)^2(\theta^-)^2(\bar{\theta}^+)^2\partial^\mu\partial_\mu M(x_L)\,. \end{aligned} \tag{2.32}$$

In the chiral basis, $\frac{1}{n!}P^n$ for $n = 1, 2, 3, 4$ in (2.26) become

$$P = -\frac{1}{2}\overleftarrow{\frac{\partial}{\partial\theta_i^\alpha}}C_{ij}^{\alpha\beta}\overrightarrow{\frac{\partial}{\partial\theta_j^\beta}}, \tag{2.33}$$

$$\begin{aligned}\frac{1}{2}P^2 =\; & \overleftarrow{\frac{\partial}{\partial\theta_1\theta_1}}\left(-\det C_{11}\overrightarrow{\frac{\partial}{\partial\theta_1\theta_1}} - \det C_{12}\overrightarrow{\frac{\partial}{\partial\theta_2\theta_2}} - \frac{1}{2}(C_{11}\;\varepsilon C_{12})^{\beta\delta}\overrightarrow{\frac{\partial}{\partial\theta_2^\delta}}\overrightarrow{\frac{\partial}{\partial\theta_1^\beta}}\right) + \\ & + \overleftarrow{\frac{\partial}{\partial\theta_2\theta_2}}\left(-\det C_{21}\overrightarrow{\frac{\partial}{\partial\theta_1\theta_1}} - \det C_{22}\overrightarrow{\frac{\partial}{\partial\theta_2\theta_2}} + \frac{1}{2}(C_{22}\;\varepsilon C_{21})^{\delta\beta}\overrightarrow{\frac{\partial}{\partial\theta_2^\delta}}\overrightarrow{\frac{\partial}{\partial\theta_1^\beta}}\right) + \\ & + \overleftarrow{\frac{\partial}{\partial\theta_1^\alpha}}\overleftarrow{\frac{\partial}{\partial\theta_2^\gamma}}\left(-\frac{1}{2}(C_{21}\;\varepsilon C_{11})^{\gamma\alpha}\overrightarrow{\frac{\partial}{\partial\theta_1\theta_1}} + \frac{1}{2}(C_{12}\;\varepsilon C_{22})^{\alpha\gamma}\overrightarrow{\frac{\partial}{\partial\theta_2\theta_2}} + \right. \\ & \qquad \left. + \frac{1}{4}C_{11}^{\alpha\beta}C_{22}^{\gamma\delta}\overrightarrow{\frac{\partial}{\partial\theta_2^\delta}}\overrightarrow{\frac{\partial}{\partial\theta_1^\beta}} + \frac{1}{4}C_{12}^{\alpha\beta}C_{21}^{\gamma\delta}\overrightarrow{\frac{\partial}{\partial\theta_1^\delta}}\overrightarrow{\frac{\partial}{\partial\theta_2^\beta}}\right), \end{aligned} \tag{2.34}$$

$$\begin{aligned}\frac{1}{3!}P^3 =\; & \frac{1}{2}\left[\left(\det C_{11}\;C_{22}^{\alpha\beta} + (C_{21}\;\varepsilon C_{11}\;\varepsilon C_{12})^{\alpha\beta}\right)\overleftarrow{\frac{\partial}{\partial\theta_1\theta_1}}\overleftarrow{\frac{\partial}{\partial\theta_2^\alpha}}\overrightarrow{\frac{\partial}{\partial\theta_2^\beta}}\overrightarrow{\frac{\partial}{\partial\theta_1\theta_1}} + \right. \\ & + \left(\det C_{12}\;C_{21}^{\alpha\beta} + (C_{22}\;\varepsilon C_{21}\;\varepsilon C_{11})^{\alpha\beta}\right)\overleftarrow{\frac{\partial}{\partial\theta_1\theta_1}}\overleftarrow{\frac{\partial}{\partial\theta_2^\alpha}}\overrightarrow{\frac{\partial}{\partial\theta_1^\beta}}\overrightarrow{\frac{\partial}{\partial\theta_2\theta_2}} + \\ & + \left(\det C_{21}\;C_{12}^{\alpha\beta} + (C_{11}\;\varepsilon C_{12}\;\varepsilon C_{22})^{\alpha\beta}\right)\overleftarrow{\frac{\partial}{\partial\theta_2\theta_2}}\overleftarrow{\frac{\partial}{\partial\theta_1^\alpha}}\overrightarrow{\frac{\partial}{\partial\theta_2^\beta}}\overrightarrow{\frac{\partial}{\partial\theta_1\theta_1}} + \\ & \left. + \left(\det C_{22}\;C_{11}^{\alpha\beta} + (C_{12}\;\varepsilon C_{22}\;\varepsilon C_{21})^{\alpha\beta}\right)\overleftarrow{\frac{\partial}{\partial\theta_2\theta_2}}\overleftarrow{\frac{\partial}{\partial\theta_1^\alpha}}\overrightarrow{\frac{\partial}{\partial\theta_1^\beta}}\overrightarrow{\frac{\partial}{\partial\theta_2\theta_2}}\right], \end{aligned} \tag{2.35}$$

$$\begin{aligned}\frac{1}{4!}P^4 =\; & \left[\det C_{11}\;\det C_{22} + (\det C_{12})^2 - (\varepsilon C_{11}\;\varepsilon C_{12}\;\varepsilon C_{22}\;\varepsilon C_{21})_\alpha{}^\alpha\right] \times \\ & \times \overleftarrow{\frac{\partial}{\partial\theta_1\theta_1}}\overleftarrow{\frac{\partial}{\partial\theta_2\theta_2}}\;\overrightarrow{\frac{\partial}{\partial\theta_2\theta_2}}\overrightarrow{\frac{\partial}{\partial\theta_1\theta_1}}\,. \end{aligned} \tag{2.36}$$

Here $(\varepsilon C)_\alpha{}^\beta \equiv \varepsilon_{\alpha\gamma}C^{\gamma\beta}$ and

$$\frac{\partial}{\partial\theta_i\theta_i} \equiv \frac{1}{4}\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta_i^\alpha}\frac{\partial}{\partial\theta_i^\beta}\,, \tag{2.37}$$

for $i = 1, 2$. Using the Poisson bracket in the chiral basis, we obtain for example,

$$\begin{aligned}
\theta_1^{+\alpha} * (\theta_2^+)^2 \theta_2^{-\beta} &= \frac{1}{2}\varepsilon^{\alpha\beta}(u_2^+ u_1^+)\theta^4 + (C_{12}^{++}\theta_2^+)^\alpha \theta_2^{-\beta} + \frac{1}{2} C_{12}^{+-\alpha\beta}(\theta_2^+)^2 , \\
\theta_1^{+\alpha}\theta_1^{-\beta} * \theta_2^{+\gamma}\theta_2^{-\delta} &= \theta_1^{+\alpha}\theta_1^{-\beta}\theta_2^{+\gamma}\theta_2^{-\delta} + \\
&\quad + \frac{1}{2}\Big(- C_{12}^{++\alpha\gamma}\theta_1^{-\beta}\theta_2^{-\delta} + C_{12}^{+-\alpha\delta}\theta_1^{-\beta}\theta_2^{+\gamma} + \\
&\qquad + C_{12}^{-+\beta\gamma}\theta_1^{+\alpha}\theta_2^{-\delta} - C_{12}^{--\beta\delta}\theta_1^{+\alpha}\theta_2^{+\gamma}\Big) + \\
&\quad + \frac{1}{4}\left(- C_{12}^{++\alpha\gamma} C_{12}^{--\beta\delta} + C_{12}^{+-\alpha\delta} C_{12}^{-+\beta\gamma}\right)
\end{aligned} \tag{2.38}$$

etc.

3. $\mathcal{N} = 2$ supersymmetric U(1) gauge theory

In this section, we study the $\mathcal{N} = 2$ supersymmetric U(1) gauge theory on the noncommutative harmonic superspace.

The action of $\mathcal{N} = 2$ super Yang-Mill theory on the harmonic superspace has been constructed in [15, 13]. The gauge superfield $V^{++}(x_A, \theta^+, \bar{\theta}^+, u)$ is introduced by gauging the $\mathcal{N} = 2$ matter action. The covariant derivative $\nabla^{++} = D^{++} + iV^{++}$ satisfies

$$\nabla^{++} \to e^{i\Lambda}\nabla^{++}e^{-i\Lambda} \tag{3.1}$$

for the analytic superfield Λ. V^{++} and Λ take values in the Lie algebra of the gauge group. The gauge transformation for V^{++} becomes

$$\delta V^{++} = -D^{++}\Lambda + i[\Lambda, V^{++}] \tag{3.2}$$

in the infinitesimal form.[1] Using the gauge superfield V^{++} the action of $\mathcal{N} = 2$ super Yang-Mills theory is given by [15, 16]

$$S = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\int d^4x d^8\theta du_1 \cdots du_n \frac{\mathrm{tr}V^{++}(\zeta_1, u_1)\cdots V^{++}(\zeta_n, u_n)}{(u_1^+ u_2^+)\cdots(u_n^+ u_1^+)} \tag{3.3}$$

where $\zeta_i = (x_A, \theta_i^+, \bar{\theta}_i^+)$ and $d^8\theta = d^4\theta^+ d^4\theta^-$ with $d^4\theta^\pm = d^2\theta^\pm d^2\bar{\theta}^\pm$. The harmonic integral is defined by the rules:

(i)

$$\int du f(u) = 0 \tag{3.4}$$

for $f(u)$ with non-zero U(1) charge.

(ii)

$$\int du 1 = 1 . \tag{3.5}$$

[1] We note that V^{++} obeys a different gauge transformation from the vector superfield V in $\mathcal{N} = 1$ gauge theory in which e^V transforms canonically.

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(iii)

$$\int du u^+_{(i_1} \cdots u^+_{i_n} u^-_{j_1} \cdots u^-_{j_n)} = 0\,, \qquad (n \geq 1)\,. \tag{3.6}$$

Since the gauge superfield contain infinite number of (auxiliary) fields, it is not convenient to discuss the off-shell theory in the component formalism. Fortunately the gauge parameters also contain infinite number of fields. We can eliminate unnecessary fields and take the Wess-Zumino(WZ) gauge:

$$\begin{aligned} V^{++}_{WZ}(x_A, \theta^+, \bar{\theta}^+, u) \;=\; & -i\sqrt{2}(\theta^+)^2\bar{\phi}(x_A) + i\sqrt{2}(\bar{\theta}^+)^2\phi(x_A) - 2i\theta^+\sigma^\mu\bar{\theta}^+A_\mu(x_A) + \\ & +4(\bar{\theta}^+)^2\theta^+\psi^i(x_A)u^-_i - 4(\theta^+)^2\bar{\theta}^+\bar{\psi}^i(x_A)u^-_i + \\ & +3(\theta^+)^2(\bar{\theta}^+)^2D^{ij}(x_A)u^-_iu^-_j\,. \end{aligned} \tag{3.7}$$

The gauge symmetry with the superfield $\Lambda(x_A)$ remains, which corresponds to the canonical gauge transformation for component fields. In the WZ gauge the sum in the action ends up to the fourth order. The action in the component formalism leads to the well-known $\mathcal{N} = 2$ action.

In the present paper, we will consider the simplest case i.e. U(1) gauge group. The gauge transformation (3.2) is

$$\delta V^{++} = -D^{++}\Lambda \tag{3.8}$$

and the action is quadratic in V^{++}_{WZ}

$$S = \frac{1}{4}\int d^4x d^8\theta du_1 du_2 \frac{V^{++}_{WZ}(x_A, \theta^+_1, \bar{\theta}^+_1, u_1)V^{++}_{WZ}(x_A, \theta^+_2, \bar{\theta}^+_2, u_2)}{(u^+_1u^+_2)^2}\,. \tag{3.9}$$

In terms of component fields, the action becomes

$$S = \int d^4x \left\{ \frac{1}{4}D^{ij}D_{ij} - i\psi^i\sigma^\mu\partial_\mu\bar{\psi}_i - \partial^\mu\phi\partial_\mu\bar{\phi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} \right\} \tag{3.10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$.

We now study the U(1) gauge theory on the noncommutative harmonic superspace by replacing the product by the $*$-product. One finds that that the quadratic action (3.9) is not invariant under the ordinary gauge transformation (3.8) due to the deformation parameter. In fact, as discussed in [9, 10], the gauge invariance of $\mathcal{N} = 2$ matter lagrangian leads to the gauge transformation (3.8)

$$\delta^* V^{++} = -D^{++}\Lambda + i[\Lambda, V^{++}]_*\,. \tag{3.11}$$

We find that the action which is invariant under the gauge transformation (3.11) is given by[2]

$$S_* = \frac{1}{2}\sum_{n=2}^{\infty}\int d^4x d^8\theta du_1 \ldots du_n \frac{(-i)^n}{n} \frac{V^{++}(1) * \cdots * V^{++}(n)}{(u^+_1u^+_2)\cdots(u^+_nu^+_1)}\,, \tag{3.12}$$

[2] $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in $\mathcal{N} = 2$ harmonic superspace with spacetime noncommutativity has been studied in [17].

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where $V^{++}(i) = V^{++}(\zeta_i, u_i)$. The gauge invariance of the action is shown in the same way as in [13]. In order to prove this, we notice that

$$\int d^4x d^4\theta \; A * B = \int d^4x d^4\theta \; AB \tag{3.13}$$

holds in the chiral-superspace integral, where $d^4\theta = d^2\theta^+ d^2\theta^-$. Then in the abelian case, we are allowed to use the formula

$$\int d^4x d^4\theta \; A * B * C = \int d^4x d^4\theta \; B * C * A \,. \tag{3.14}$$

Together with the cyclic property of the denominator $(u_1^+u_2^+)(u_2^+u_3^+)\cdots(u_{n-1}^+u_n^+)(u_n^+u_1^+)$ in eq. (3.12), the gauge variation of the action can be written as

$$\begin{aligned} \delta^* S_* &= \frac{1}{2}\sum_{n=2}^{\infty}\int d^4x d^8\theta du_1 \ldots du_n (-i)^n \Big(\delta^* V^{++}(1)\Big) \frac{V^{++}(2) * \cdots * V^{++}(n)}{(u_1^+u_2^+)\cdots(u_n^+u_1^+)} \\ &= \frac{1}{2}\int d^4x d^8\theta du_1 \Big(\delta^* V^{++}(X, u_1)\Big) V_*^{--}(X, u_1) \end{aligned} \tag{3.15}$$

where

$$V_*^{--}(X, u) \equiv \sum_{n=1}^{\infty}\int du_1 \ldots du_n (-i)^{n+1} \frac{V^{++}(1) * \cdots * V^{++}(n)}{(u^+u_1^+)(u_1^+u_2^+)\cdots(u_n^+u_1^+)(u_1^+u^+)} \,. \tag{3.16}$$

V_*^{--} satisfies the relation

$$D^{++}V_*^{--} - D^{--}V^{++} + i[V_*^{--}, V^{++}]_* = 0 \,. \tag{3.17}$$

Substituting $\delta^* V^{++}$ given in eq. (3.11) into $\delta^* S_*$, integrating by parts, with the use of the above relation, we find

$$\delta^* S_* = \frac{1}{2}\int d^4x d^8\theta \int du \; \Lambda D^{--}V_*^{++} \,. \tag{3.18}$$

As in the non-abelian case [13], the analyticity of the gauge parameter Λ ensures that this integral vanishes.

In the WZ gauge, the action becomes

$$S_* = \frac{1}{2}\sum_{n=2}^{\infty}\int d^4x d^8\theta du_1 \ldots du_n \frac{(-i)^n}{n} \frac{V_{WZ}^{++}(1) * \cdots * V_{WZ}^{++}(n)}{(u_1^+u_2^+)\cdots(u_n^+u_1^+)} \,. \tag{3.19}$$

In the following we examine the action in terms of component fields. In the present work, however, we will calculate the action explicitly up to the third order in V_{WZ}^{++}. Although it is possible to calculate higher order terms, explicit calculation is very complicated due to the harmonic integrals, which is left for future works.

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The terms coming from the second order of V^{++}_{WZ} are the same as the commutative gauge theory (3.10) due to eq. (3.13). The second order action $S_{*,2}$ is given by

$$S_{*,2} \equiv \frac{1}{4}\int d^4x d^8\theta du_1 du_2 \frac{V^{++}_{WZ}(1) * V^{++}_{WZ}(2)}{(u_1^+ u_2^+)^2}$$
$$= \int d^4x \left[-\frac{1}{4}F_{\mu\nu}(F^{\mu\nu} + \tilde{F}^{\mu\nu}) - i\psi^i\sigma^\mu\partial_\mu\bar{\psi}_i + \phi\partial^\mu\partial_\mu\bar{\phi} + \frac{1}{4}D_{ij}D^{ij} \right]. \qquad (3.20)$$

The second order terms do not include the deformation parameter. On the other hand, in the abelian case, all the terms coming from the third order of V^{++}_{WZ}, which we will denote $S_{*,3}$, contain the deformation parameter C. They are all the newly emerged terms due to the deformation. This is true for the terms coming from the higher order contributions in terms of V^{++}_{WZ}.

Note, however, that the $\mathcal{O}(C^2)$ and $\mathcal{O}(C^4)$-terms in $S_{*,3}$ vanish: for Grassmann even functions A_1 and A_2, the equation

$$A_1 P^n A_2 = (-1)^n A_2 P^n A_1 \qquad (3.21)$$

holds, while the harmonic distribution $(u_1^+u_2^+)^{-1}(u_2^+u_3^+)^{-1}(u_3^+u_1^+)^{-1}$ is completely anti-symmetric under the permutations of (u_1, u_2, u_3).

In addition to this, when we calculate the terms in $S_{*,3}$ it is useful to notice a relation between distinct contributions, which is given below. In the chiral-superspace integral, because of eq. (3.13), it holds that

$$\int d^4x d^4\theta \; A_1 * A_2 * A_3 = \int d^4x d^4\theta \; (A_1 * A_2) \; A_3 = \int d^4x d^4\theta \; A_1 \; (A_2 * A_3) \qquad (3.22)$$

up to total derivatives, so that

$$\int d^4x d^4\theta \; (A_1 P A_2) \; A_3 = \int d^4x d^4\theta \; A_1 \; (A_2 P A_3) \qquad (3.23)$$

also holds. This shows that these potentially distinct contributions are the same. Moreover, in the abelian case, together with eq. (3.21), we have

$$\int d^4x d^8\theta \frac{du_1 du_2 du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \; \frac{1}{6!}\sum_{\sigma\in S_3} \left(A_{\sigma(1)}(u_1) P A_{\sigma(2)}(u_2)\right)\right) \; A_{\sigma(3)}(u_3) =$$
$$= \int d^4x d^8\theta \frac{du_1 du_2 du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \left(A_1(u_1) P A_2(u_2)\right) \; A_3(u_3)\,, \qquad (3.24)$$

where S_3 is the permutation group of degree 3. Thus, among the terms in the left hand side we are allowed to choose the most simple combination to calculate.

The terms coming from the third order of V^{++}_{WZ} are followings:

$$S_{*,3} \equiv \frac{i}{6}\int d^4x d^8\theta du_1 du_2 du_3 \frac{V^{++}_{WZ}(1) * V^{++}_{WZ}(2) * V^{++}_{WZ}(3)}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}$$
$$= \int d^4x \Big[+ \frac{1}{\sqrt{2}} C_s A_\nu \partial_\mu \bar{\phi}(F^{\mu\nu} + \tilde{F}^{\mu\nu}) + \frac{i}{\sqrt{2}} C_s \bar{\phi}(\psi^k\sigma^\nu\partial_\nu\bar{\psi}_k) +$$

$$
\begin{aligned}
&- \frac{2\sqrt{2}}{3} i C^{\alpha\beta}_{(ij)} \psi^i_\alpha (\sigma^\nu \partial_\nu \bar\psi^j)_\beta \bar\phi + \frac{i}{\sqrt{2}} C_s (\psi^k \sigma^\nu \bar\psi_k) \partial_\nu \bar\phi - \\
&- 2\sqrt{2} i C^{\alpha\beta}_{(ij)} \psi^i_\alpha (\sigma^\nu \bar\psi^j)_\beta \partial_\nu \bar\phi - \frac{i}{2} C_s \varepsilon^{\alpha\beta} A_\mu (\sigma^\mu \bar\psi^k)_\alpha (\sigma^\nu \partial_\nu \bar\psi_k)_\beta + \\
&+ \frac{2}{3} i C^{\alpha\beta}_{(ij)} A_\mu (\sigma^\mu \bar\psi^i)_\alpha (\sigma^\nu \partial_\nu \bar\psi^j)_\beta + \frac{i}{2} C_s \bar\psi^i \bar\psi^j D_{ij} - i C^{\mu\nu}_{(ij)} \bar\psi^i \bar\psi^j F_{\mu\nu} + \\
&+ \frac{\sqrt{2}}{4} C_s A_\mu A^\mu \partial^2 \bar\phi - \frac{\sqrt{2}}{4} C_s \bar\phi D^{ij} D_{ij} + \\
&+ \sqrt{2} C^{\mu\nu}_{(ij)} D^{ij} A_\mu \partial_\nu \bar\phi + \frac{1}{\sqrt{2}} C^{\mu\nu}_{(ij)} D^{ij} F_{\mu\nu} \bar\phi \Bigg] . \qquad (3.25)
\end{aligned}
$$

Detailed calculations of these terms are explained in appendix B.

At first sight there are terms which are not gauge invariant in the sense of the ordinary gauge transformation. Of course the deformed gauge variation $\delta^* V^{++}$ of the gauge superfield contains the deformation parameter C, so that $\delta^* S_*$ also depends on C. As the gauge invariance of S_* has been shown in the superfield formalism, the $\mathcal{O}(C^1)$ part of $\delta^* S_{*,2}$, for example, should cancel with the $\mathcal{O}(C^1)$ part of $\delta^* S_{*,3}$. In the following, we would like to see how such a cancellation is actually realized in the component action in the WZ gauge.

4. The deformed gauge transformation

In this section we study the gauge transformation of $\mathcal{N} = 2$ supersymmetric U(1) gauge theory on the noncommutative harmonic superspace.

The deformed gauge transformation of the gauge superfield in the WZ gauge, V^{++}_{WZ}, with an analytic gauge parameter $\lambda(\zeta, u)$ ($\zeta = (x_A, \theta^+, \bar\theta^+)$) is given by

$$
\delta^*_\lambda V^{++}_{WZ}(\zeta, u) = -D^{++}_A \lambda(\zeta, u) + i[\lambda, V^{++}_{WZ}]_*(\zeta, u) . \qquad (4.1)
$$

If we choose the gauge parameter $\lambda(\zeta, u) = \lambda(x_A)$ as in the commutative case, the corresponding gauge variation becomes

$$
\begin{aligned}
\delta^*_\lambda V^{++}_{WZ}(\zeta, u) \;=\; & -2i\, \theta^+ \sigma^\mu \bar\theta^+ \left(-\partial_\mu \lambda - \frac{1}{\sqrt{2}} C_s \partial_\mu \lambda \bar\phi + 2\sqrt{2} \eta_{\mu\rho} C^{(+-)\rho\sigma} \partial_\sigma \lambda \bar\phi \right)(x_A) + \\
& + \sqrt{2} i (\bar\theta^+)^2 \left(\frac{1}{\sqrt{2}} C_s \partial_\mu \lambda A^\mu + 2\sqrt{2} C^{(+-)\mu\nu} \partial_\mu \lambda A_\nu \right)(x_A) + \\
& + 4 (\bar\theta^+)^2 \theta^{+\alpha} \left(-\frac{1}{2} C_s \partial_\mu \lambda (\sigma^\mu \bar\psi^-)_\alpha - 2 \partial_\mu \lambda (\varepsilon C^{(+-)} \sigma^\mu \bar\psi^-)_\alpha \right)(x_A) + \\
& + 3 (\theta^+)^2 (\bar\theta^+)^2 \left(\frac{4\sqrt{2}}{3} C^{--\mu\nu} \partial_\mu \lambda \partial_\nu \bar\phi \right)(x_A) , \qquad (4.2)
\end{aligned}
$$

where $C^{(+-)\alpha\beta} \equiv C^{\alpha\beta}_{(ij)} u^{+i} u^{-j}$. This transformation obviously does not preserve the WZ gauge because each θ-components in (4.2) contain extra u-dependent terms. Therefore we need to regard the gauge parameter as more general analytic superfield. If we choose an analytic gauge parameter of the form

$$
\begin{aligned}
\lambda_C(\zeta, u; C) \;=\; & \lambda(x_A) + \theta^+ \sigma^\mu \bar\theta^+ \lambda^{(-2)}_\mu (x_A, u; C) + (\bar\theta^+)^2 \lambda^{(-2)}(x_A, u; C) + \\
& + (\bar\theta^+)^2 \theta^{+\alpha} \lambda^{(-3)}_\alpha (x_A, u; C) + (\theta^+)^2 (\bar\theta^+)^2 \lambda^{(-4)}(x_A, u; C) , \qquad (4.3)
\end{aligned}
$$

then the corresponding gauge variation is

$$
\begin{aligned}
\delta^*_{\lambda_C} V^{++}_{WZ}(\zeta,u) = {} & \theta^+\sigma^\mu\bar\theta^+\Big(+ 2i\partial_\mu\lambda + \sqrt{2}iC_s\partial_\mu\lambda\ \bar\phi + 2\sqrt{2}iC^{(+-)\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ \bar\phi - \\
& \qquad - \partial^{++}\lambda^{(-2)}_\mu - \sqrt{2}C^{++\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\lambda^{(-2)}_\nu\bar\phi\Big)(x_A) + \\
& + (\bar\theta^+)^2\Big(iC_s\partial_\mu\lambda\ A^\mu - 2iC^{(+-)\alpha\beta}(\sigma^\nu\bar\sigma^\mu\varepsilon)_{\alpha\beta}\partial_\mu\lambda\ A_\nu + \\
& \qquad + C^{++\alpha\beta}(\sigma^\nu\bar\sigma^\mu\varepsilon)_{\alpha\beta}\lambda^{(-2)}_\mu A_\nu - \partial^{++}\lambda^{(-2)}\Big)(x_A) + \\
& + (\bar\theta^+)^2\theta^{+\alpha}\Big(- 2C_s\partial_\mu\lambda\ (\sigma^\mu\bar\psi^-)_\alpha - 4(\sigma^\mu\bar\psi^-)_\alpha C^{(+-)\gamma\delta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\gamma\delta}\partial_\nu\lambda - \\
& \qquad - 2i(\sigma^\mu\bar\psi^-)_\alpha C^{++\gamma\delta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\gamma\delta}\lambda^{(-2)}_\nu - \\
& \qquad - \partial^{++}\lambda^{(-3)}_\alpha - 2\sqrt{2}(\varepsilon C^{++}\lambda^{(-3)})_\alpha\bar\phi\Big)(x_A) + \\
& + (\theta^+)^2(\bar\theta^+)^2\Big(4\sqrt{2}C^{--\mu\nu}\partial_\mu\lambda\ \partial_\nu\bar\phi - i\partial^\mu\lambda^{(-2)}_\mu - \sqrt{2}iC^{+-\alpha\beta}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\times \\
& \qquad \times\partial_\nu(\lambda^{(-2)}_\mu\bar\phi) - \partial^{++}\lambda^{(-4)}\Big)(x_A)\,,
\end{aligned}
\tag{4.4}
$$

where $(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta} = (\sigma^\mu\bar\sigma^\nu)_\alpha{}^\gamma\varepsilon_{\gamma\beta}$. Note that other θ components will never emerge as long as we use the gauge parameter λ_C of the form given above. Moreover, the degrees of freedom contained in $\lambda^{(-2)}_\mu$, $\lambda^{(-2)}$, $\lambda^{(-3)}_\alpha$ and $\lambda^{(-4)}$ are not only necessary but also sufficient to consistently gauge away all of the terms with extra u-dependence. Therefore, with an appropriate tuning of the analytic gauge parameter λ_C, this variation can be brought into the form

$$
\begin{aligned}
\delta^*_{\lambda_C} V^{++}_{WZ}(\zeta,u) = {} & -2i\ \theta^+\sigma^\mu\bar\theta^+\left(\delta^*_{\lambda_C}A_\mu(x_A)\right) + \sqrt{2}i\ (\bar\theta^+)^2\left(\delta^*_{\lambda_C}\phi(x_A)\right) + \\
& + 4\ (\bar\theta^+)^2\theta^{+\alpha}\left(\delta^*_{\lambda_C}\psi^i_\alpha(x_A)u^-_i\right) + 3\ (\theta^+)^2(\bar\theta^+)^2\left(\delta^*_{\lambda_C}D^{ij}(x_A)u^-_iu^-_j\right),
\end{aligned}
\tag{4.5}
$$

where $\delta^*_{\lambda_C}A_\mu$, $\delta^*_{\lambda_C}\phi$, $\delta^*_{\lambda_C}\psi^i_\alpha$ and $\delta^*_{\lambda_C}D^{ij}$ are u-independent quantities, so that each gives the proper gauge transformation law for the respective component fields in this deformed theory.

In the following, we will determine $\delta^*_{\lambda_C}A_\mu$, $\delta^*_{\lambda_C}\phi$ and so on.

4.1 The deformed gauge transformation of the gauge field

First of all, we will concentrate on the $\theta^+\bar\theta^+$-component of the gauge variation $\delta^*_{\lambda_C}V^{++}_{WZ}$ to determine a properly deformed gauge transformation of the gauge field, $\delta^*_{\lambda_C}A_\mu$, and the correspondingly deformed component of the gauge parameter, $\lambda^{(-2)}_\mu$. This is because the other components depend on $\lambda^{(-2)}_\mu$. The rest of the components will be determined in much the same way.

Comparing the $\theta^+\bar\theta^+$-components of eqs. (4.4) and (4.5), we find the equation which determines $\delta^*_{\lambda_C}A_\mu$ and $\lambda^{(-2)}_\mu$:

$$
\begin{aligned}
-2i\delta^*_{\lambda_C}A_\mu = {} & 2i\partial_\mu\lambda + \sqrt{2}iC_s\partial_\mu\lambda\ \bar\phi + 2\sqrt{2}iC^{(+-)\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ \bar\phi - \\
& - \partial^{++}\lambda^{(-2)}_\mu - \sqrt{2}C^{++\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\lambda^{(-2)}_\nu\bar\phi,
\end{aligned}
\tag{4.6}
$$

where $\delta^*_{\lambda_C} A_\mu$ is identified with the lowest order term in the harmonic expansion of the left hand side (i.e. the U(1) charge 0, isospin 0 part). Note that the gauge parameters $\lambda_\mu^{(-2)}$ admit not only the harmonic expansion but also the C-expansion at the same time, while $\delta^*_{\lambda_C} A_\mu$ can be expanded only in terms of C by its definition. Here we first formally expand both of them in terms of C,

$$\delta^*_{\lambda_C} A_\mu \equiv \sum_{N=0}^{\infty} (\delta^*_{\lambda_C} A_\mu)_{(N)} \,, \tag{4.7}$$

$$\lambda_\mu^{(-2)} \equiv \sum_{N=1}^{\infty} \lambda_\mu^{(-2)}{}_{(N)} \,, \tag{4.8}$$

where $(\delta^*_{\lambda_C} A_\mu)_{(N)}$ and $\lambda_\mu^{(-2)}{}_{(N)}$ are $\mathcal{O}(C^N)$-quantities. Note also that the C-independent part of $\lambda_\mu^{(-2)}$ is absent, because in the $C \to 0$ limit this should vanish. Substituting these into eq. (4.6), we find the following equations:

$$(\delta^*_{\lambda_C} A_\mu)_{(0)} = -\partial_\mu \lambda \,, \tag{4.9}$$

$$(\delta^*_{\lambda_C} A_\mu)_{(1)} = -\frac{1}{\sqrt{2}} C_s \partial_\mu \lambda \, \bar{\phi} \,, \tag{4.10}$$

$$\partial^{++} \lambda_\mu^{(-2)}{}_{(1)} = 2\sqrt{2} i C^{(+-)\alpha\beta} (\sigma_\mu \bar{\sigma}^\nu \varepsilon)_{\alpha\beta} \partial_\nu \lambda \, \bar{\phi} \,, \tag{4.11}$$

$$\partial^{++} \lambda_\mu^{(-2)}{}_{(N)} = -\sqrt{2} C^{++\alpha\beta} (\sigma^\mu \bar{\sigma}^\nu \varepsilon)_{\alpha\beta} \lambda_\nu^{(-2)}{}_{(N-1)} \bar{\phi} + 2i (\delta^*_{\lambda_C} A_\mu)_{(N)} \qquad (N \geq 2) \,. \tag{4.12}$$

The first two equations are immediately read from the left hand side of eq. (4.6). They are giving the lowest and the next lowest order terms in the C-expansion of $\delta^*_{\lambda_C} A_\mu$. The rest of two equations are obtained by equating the terms of the same order in C in both sides of eq. (4.6).

The latter two equations can be regarded as a recursion relation to determine $\lambda_\mu^{(-2)}{}_{(N)}$ and $(\delta^*_{\lambda_C} A_\mu)_{(N)}$ from $\lambda_\mu^{(-2)}{}_{(N-1)}$ with an initial condition given by $\lambda_\mu^{(-2)}{}_{(1)}$. It is useful to treat explicitly the coefficient functions in the harmonic expansion of $\lambda_\mu^{(-2)}{}_{(N)}$:

$$\begin{aligned}
\lambda_\mu^{(-2)}{}_{(N)} &= \sum_{n=0}^{\infty} (\lambda_\mu^{(-2)}{}_{(N)})^{(i_1 \ldots i_n k_1 \ldots k_{n+2})} u^+_{i_1} \ldots u^+_{i_n} u^-_{k_1} \ldots u^-_{k_{n+2}} \\
&= \sum_{n=0}^{\infty} (\lambda_\mu^{(-2)}{}_{(N)})_{(i_1 \ldots i_n k_1 \ldots k_{n+2})} u^{+i_1} \ldots u^{+i_n} u^{-k_1} \ldots u^{-k_{n+2}} \,.
\end{aligned} \tag{4.13}$$

We first note that, from the general reduction identity of harmonic variables [13], the following reduction formula holds for any term in this harmonic expansion:

$$\begin{aligned}
&C^{++\alpha\beta} (\lambda_\mu^{(-2)}{}_{(N)})_{(j_{n+m} j_{n+m-1} \cdots j_1)} \overbrace{u^{+(j_{n+m}} \ldots u^{+j_{m+1}}}^{n} \overbrace{u^{-j_m} \ldots u^{-j_1)}}^{m} = \\
&= C^{\alpha\beta}_{(kl)} (\lambda_\mu^{(-2)}{}_{(N)})_{(j_{n+m} j_{n+m-1} \cdots j_1)} u^{+(k} u^{+l)} \overbrace{u^{+(j_{n+m}} \ldots u^{+j_{m+1}}}^{n} \overbrace{u^{-j_m} \ldots u^{-j_1)}}^{m} \\
&= C^{\alpha\beta}_{(kl)} (\lambda_\mu^{(-2)}{}_{(N)})_{(j_{n+m} j_{n+m-1} \cdots j_1)} \times
\end{aligned}$$

$$
\begin{aligned}
&\times\Bigg(\overbrace{u^{+(k}u^{+l}u^{+j_{n+m}}\dots u^{+j_{m+1}}}^{n+2}\overbrace{u^{-j_m}\dots u^{-j_1)}}^{m} -\\
&\quad -\frac{2m}{n+m+2}\epsilon^{lj_1}\overbrace{u^{+(k}u^{+j_{n+m}}\dots u^{+j_{m+1}}}^{n+1}\overbrace{u^{-j_m}\dots u^{-j_2)}}^{m-1} +\\
&\quad +\frac{m(m-1)}{(n+m+1)(n+m)}\epsilon^{kj_2}\epsilon^{lj_1}\overbrace{u^{+(j_{n+m}}\dots u^{+j_{m+1}}}^{n}\overbrace{u^{-j_m}\dots u^{-j_3)}}^{m-2}\Bigg). \qquad (4.14)
\end{aligned}
$$

Then eqs. (4.11) and (4.12) can be rewritten as the following equations:[3]

$$
\begin{aligned}
\lambda_\mu^{(-2)}{}_{(1)} &= (\lambda_\mu^{(-2)}{}_{(1)})_{(ij)}u^{-i}u^{-j} = \sqrt{2}iC^{\alpha\beta}_{(ij)}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\;\bar{\phi}\;u^{-i}u^{-j}\,, && (4.16)\\
(\lambda_\mu^{(-2)}{}_{(N)})_{(j_{2m}j_{2m-1}\cdots j_1)} &= \frac{1}{(2m)!}\sum_{\sigma\in S_{2m}}\\
&\quad\times\Bigg[\frac{1}{m+1}C^{\alpha\beta}_{(j_{\sigma(2m)}j_{\sigma(2m-1)})}(\lambda_\nu^{(-2)}{}_{(N-1)})_{(j_{\sigma(2m-2)}j_{\sigma(2m-3)}\cdots j_{\sigma(1)})} -\\
&\quad -\frac{1}{m+1}\epsilon^{kl}C^{\alpha\beta}_{(k\,j_{\sigma(2m)})}(\lambda_\nu^{(-2)}{}_{(N-1)})_{(j_{\sigma(2m-1)}j_{\sigma(2m-2)}\cdots j_{\sigma(1)}\,l)} +\\
&\quad +\frac{(m+2)}{(2m+2)(2m+3)}\epsilon^{km}\epsilon^{ln}C^{\alpha\beta}_{(kl)}(\lambda_\nu^{(-2)}{}_{(N-1)})_{(j_{\sigma(2m)}j_{\sigma(2m-1)}\cdots j_{\sigma(1)}mn)}\Bigg]\times\\
&\quad\times(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}(-\sqrt{2}\bar{\phi})\qquad (m\le N)\,, && (4.17)\\
(\lambda_\mu^{(-2)}{}_{(N)})_{(j_{2m}j_{2m-1}\cdots j_1)} &= 0\qquad (m>N)\,, && (4.18)\\
(\delta^*_{\lambda_C}A_\mu)_{(N)} &= -\frac{i}{3\sqrt{2}}\epsilon^{ik}\epsilon^{jl}C^{\alpha\beta}_{(ij)}(\lambda_\nu^{(-2)}{}_{(N-1)})_{(kl)}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\bar{\phi}\qquad (N\ge 2)\,. && (4.19)
\end{aligned}
$$

Here we have used the relation

$$
(\partial^{++})^{-1}\Big(\overbrace{u^+_{(j_{2m}}u^+_{j_{2m-1}}\dots u^+_{j_{m+1}}}^{m}\overbrace{u^-_{j_m}\dots u^-_{j_1)}}^{m}\Big) = \frac{1}{m+1}\overbrace{u^+_{(j_{2m}}u^+_{j_{2m-1}}\dots u^+_{j_{m+2}}}^{m-1}\overbrace{u^-_{j_{m+1}}\dots u^-_{j_1)}}^{m+1}\,. \qquad (4.20)
$$

S_{2m} is the permutation group of degree $2m$.

From a direct calculation up to the second order in C, $\lambda_\mu^{(-2)}$ is determined as

$$
\begin{aligned}
\lambda_\mu^{(-2)} &= -2\sqrt{2}iC^{--}{}_\mu{}^\nu\partial_\nu\lambda\;\bar{\phi} - 4i\left(C_{(ij)}\varepsilon C_{(kl)}\right)^{\alpha\beta}\times\\
&\quad\times(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\;\bar{\phi}^2\left(\frac{1}{3}u^{+(i}u^{-j}u^{-k}u^{-l)} - \frac{1}{2}\epsilon^{jl}u^{-(i}u^{-k)}\right) + \mathcal{O}(C^3)\,, \qquad (4.21)
\end{aligned}
$$

[3]Strictly speaking, the expression (4.17) is only valid for $m\ge 2$, because the first term in the square bracket is absent if $m=1$. Thus for $m=1$, instead of eq. (4.17), we should use

$$
\begin{aligned}
(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}u^{-i}u^{-j} &= \left[-\frac{1}{2}\epsilon^{kl}C^{\alpha\beta}_{(ik)}(\lambda_\nu^{(-2)}{}_{(N-1)})_{(jl)} + \frac{3}{20}\epsilon^{km}\epsilon^{ln}C^{\alpha\beta}_{(kl)}(\lambda_\nu^{(-2)}{}_{(N-1)})_{(ijmn)}\right]\times\\
&\quad\times(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}(-\sqrt{2}\bar{\phi})u^{-i}u^{-j}\,. \qquad (4.15)
\end{aligned}
$$

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where we have used the identity

$$(\sigma^\mu\bar{\sigma}^\rho\varepsilon)_{\alpha\beta}(\sigma_\rho\bar{\sigma}^\nu\varepsilon)_{\gamma\delta} = 2\varepsilon_{\beta\gamma}(\sigma^\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\delta}\,. \tag{4.22}$$

Based on the observation of the result from direct calculation, (4.21), we will assume $(\lambda_\mu^{(-2)}{}_{(N)})_{(j_1\ldots j_{2m})}$ ($m \le N$) has the form

$$(\lambda_\mu^{(-2)}{}_{(N)})_{(j_1\ldots j_{2m})} = \left[\sum\left(\text{const.}\times\mathcal{C}^{\alpha\beta}_{(N)(j_1\ldots j_{2m})}\right)\right](\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\;\bar{\phi}^N\,, \tag{4.23}$$

where $\mathcal{C}^{\alpha\beta}_{(N)(j_1\ldots j_{2m})}$ represents a term of the form

$$\left(C_{(j_1j_2)}\cdot C_{(j_3j_4)}\cdots C_{(j_{2N-1}j_{2N})}\right)^{\alpha\beta} \equiv C^{\alpha\alpha_2}_{(j_1j_2)}\varepsilon_{\alpha_2\alpha_3}C^{\alpha_3\alpha_4}_{(j_3j_4)}\varepsilon_{\alpha_4\alpha_5}\cdots\varepsilon_{\alpha_{2N-2}\alpha_{2N-1}}C^{\alpha_{2N-1}\beta}_{(j_{2N-1}j_{2N})} \tag{4.24}$$

with $2(N-m)$ SU(2) indices contracted by ϵ tensors and the rest of SU(2) indices completely symmetrized, which are renamed $j_1,\ldots,j_{2m}$. The summation in (4.23) is taken over the appropriate contractions of the SU(2) indices. From eq. (4.18) and (4.22), we can easily see that $(\lambda_\mu^{(-2)}{}_{(N+1)})_{(j_1\ldots j_{2m})}$ also has the same form as the assumption (4.23). $(\lambda_\mu^{(-2)}{}_{(1)})_{(ij)}$ also takes the same form as (4.16). Therefore, the assumption (4.23) is true for any N.

It is important to know $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$, because $\delta^*_{\lambda_C}A_\mu$ can be determined by them as was seen in eq. (4.19). Based on the general form (4.23) and the fact stated below, the full form of $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$ can be written down explicitly, though we will actually give a formula for $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}u^{-i}u^{-j}$. $(\lambda_\mu^{(-2)}{}_{(N-1)})_{(ij)}$ is determined by the recursion relations (4.17), (4.18) and (4.15). Eq. (4.17) tells us that any term in $(\lambda_\mu^{(-2)}{}_{(N-1)})_{(j_1j_2\ldots j_{2m})}$ ($m \ge 2$) which has $2m$ uncontracted SU(2) indices will split into three terms in $\lambda_\mu^{(-2)}{}_{(N)}$: terms with $2m+2$, $2m$ and $2m-2$ uncontracted SU(2) indices. In this way, any specific term in $(\lambda_\mu^{(-2)}{}_{(N-1)})_{(ij)}$ ($N \ge 2$) can be traced back to $(\lambda_\mu^{(-2)}{}_{(1)})_{(ij)}$ given in (4.16) through a unique track of increasing and decreasing of the number of uncontracted SU(2) indices.

Based on these facts, one of the full form of $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$ can be formulated as follows: Define a series of numbers $\{\eta_2,\eta_3,\ldots,\eta_N\}$, where

$$\eta_i = \{-1, 0, +1\} \qquad (i = 2,\ldots,N)\,, \tag{4.25}$$

satisfying

$$m_n \ge 1 \qquad \text{for } {}^\forall n \le N-1\,, \qquad m_N = 1\,. \tag{4.26}$$

Here we have defined m_n as follows:

$$m_1 \equiv 1\,, \qquad m_n \equiv 1 + \sum_{i=2}^{n}\eta_i \qquad (n \ge 2)\,. \tag{4.27}$$

Each series corresponds to "a unique track" stated above: η_n determines which term of three terms in eq. (4.17) is chosen in the given path, from the $(n-1)$-th step to the next. If η_n is $+1$, the first term in eq. (4.17) is chosen. Therefore the number of uncontracted SU(2) indices will increase by two compared to the former step. Similarly, $\eta_n = 0$ corresponds to the choice of the second term thus the number of uncontracted indices will be unchanged.

$\eta_n = -1$ corresponds to the third term and the number of indices will decrease by two. m_n denotes a half of the number of uncontracted SU(2) indices at the n-th step. The latter condition in eq. (4.26) restrict a path to the one which ends up with $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$: the number of uncontracted SU(2) indices should be two at the N-th step. With the use of the set of such series, we can write $(\lambda_\mu^{(-2)})_{(ij)} \equiv \sum_N (\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$ as

$$(\lambda_\mu^{(-2)})_{(ij)}\ u^{-i} u^{-j} = -h^{--}{}_\mu{}^\nu(\bar\phi)\ \partial_\nu \lambda \,. \tag{4.28}$$

Here we have defined

$$\begin{aligned} h^{--}{}_\mu{}^\nu(\bar\phi) &= h_{(ij)\mu}{}^\nu(\bar\phi)\ u^{-i}u^{-j} \\ &\equiv 2\sqrt{2} i C^{--}{}_\mu{}^\nu \bar\phi \\ &\quad + \frac{i}{2} \sum_{N=2}^{\infty} \left(C_{(j_{2N} j_{2N-1})} \cdot C_{(j_{2N-2} j_{2N-3})} \cdots C_{(j_2 j_1)}\right)^{\alpha\beta} \times \\ &\quad \times (\sigma_\mu \bar\sigma^\nu \varepsilon)_{\alpha\beta} (-2\sqrt{2}\bar\phi)^N \sum_{\{\eta_2,\ldots,\eta_N\}} \frac{1}{(2m_1)!} \cdots \frac{1}{(2m_N)!} \sum_{\sigma_2} \cdots \sum_{\sigma_{N+1}} \\ &\quad \times \left[\prod_{\{n|\eta_n=+1\}} \alpha_n\right] \left[\prod_{\{n|\eta_n=0\}} \beta_n \epsilon^{j_{2n-1} j_{\sigma_n(k_n)}}\right] \times \\ &\quad \times \left[\prod_{\{n|\eta_n=-1\}} \gamma_n \epsilon^{j_{2n} j_{\sigma_n(l_n)}} \epsilon^{j_{2n-1} j_{\sigma_n(k_n)}}\right] u^{-j_{\sigma_{N+1}(l_{N+1})}} u^{-j_{\sigma_{N+1}(k_{N+1})}} \,, \end{aligned} \tag{4.29}$$

where

$$\alpha_n \equiv \frac{1}{m_{n-1}+2} \,, \tag{4.30}$$

$$\beta_n \equiv -\frac{1}{m_{n-1}+1} \,, \tag{4.31}$$

$$\gamma_n \equiv \frac{m_{n-1}+1}{2m_{n-1}(2m_{n-1}+1)} \,. \tag{4.32}$$

σ_n means a permutation of the elements of $I_n \setminus K_n$, where

$$I \equiv \left\{1,2,\ldots,2N\right\} \setminus \left\{2i-1, 2j-1, 2j \,\middle|\, {}^\forall i, \eta_i = 0; {}^\forall j, \eta_j = -1\right\}, \tag{4.33}$$

$$I_1 \equiv \emptyset \,, \qquad I_n \equiv \left\{i \,\middle|\, {}^\forall i \le 2n-2, i \in I\right\} (2 \le n \le N+1) \,, \tag{4.34}$$

$$\begin{aligned} K_1, K_2 &\equiv \emptyset \,, \\ K_n &\equiv \left\{\sigma_p(k_p), \sigma_q(k_q), \sigma_q(l_q) \,\middle|\, {}^\forall p < n, \eta_p = 0, {}^\forall q < n, \eta_q = -1\right\} \\ &\qquad (3 \le n \le N+1) \,, \end{aligned} \tag{4.35}$$

$$k_n \equiv \min(I_n \setminus K_n)\ (n \ge 2) \,, \tag{4.36}$$

$$l_n \equiv \min(I_n \setminus K_n \setminus \{k_n\})\ (n \ge 2) \,. \tag{4.37}$$

As was seen from eq. (4.17) together with the definition of η_n, from the $(n-1)$-th step to the n-th step, $(C \cdots C)^{\alpha\beta}$ is multiplied by $C_{(ij)}$ from the left and they will become $(C_{(ij)} \cdot$

$C\cdots C)^{\alpha\beta}$, while the number of uncontracted SU(2) indices are not necessarily increased by two: indices of $C_{(ij)}$ may be contracted with uncontracted indices of $(C\cdots C)^{\alpha\beta}$ according to η_n (zero for $\eta_n = +1$, one for $\eta_n = 0$ and two for $\eta_n = -1$). Thus which of the $2N$ indices are contracted with an index in the right side is determined uniquely by the series $\{\eta_2, \ldots, \eta_N\}$ (up to the permutation of two indices of each C). I is a set of the labels of the remaining indices. I_n is a subset of I, where the indices are restricted to the ones placed at $n-1$ C's from the right, that is, $(C\cdots C)^{\alpha\beta}$ above. In general, some of the indices specified by I_n have already contracted at the n-th step and only m_n indices of I_n are uncontracted. K_n specifies the indices of I_n which have already contracted by ϵ tensors. Thus $I_n \setminus K_n$ specifies the m_n uncontracted SU(2) indices at the $(n-1)$-th step, which should be completely symmetrized. k_n (and also l_n) has a meaning of the rightmost one of those indices (of course instead we can choose the leftmost one or another specific index because they are symmetrized). $h^{--}{}_\mu{}^\nu(\bar\phi)$ is expanded as

$$h^{--}{}_\mu{}^\nu(\bar\phi) = 2\sqrt{2}iC^{--}{}_\mu{}^\nu\bar\phi - 4i\epsilon^{mn}\left(C_{(im)}\cdot C_{(jn)}\right)^{\alpha\beta}(\sigma_\mu{}^\nu\varepsilon)_{\alpha\beta}\bar\phi^2\ u^{-i}u^{-j} + \mathcal{O}(C^3)\,. \quad (4.38)$$

If we use the formula (4.28), eq. (4.19) tells us that one of the full form of $\delta^*_{\lambda_C}A_\mu$ can be formulated as

$$\delta^*_{\lambda_C}A_\mu = -\partial_\mu\lambda - f(\bar\phi)\bar\phi\ \partial_\mu\lambda\,. \quad (4.39)$$

Here we have defined

$$\begin{aligned} f(\bar\phi) \equiv{}& \frac{1}{\sqrt{2}}C_s - \frac{1}{4}\sum_{N=2}^{\infty}\mathrm{Tr}\left[C_{(j_{2N}j_{2N-1})}\cdot C_{(j_{2N-2}j_{2N-3})}\cdots C_{(j_2j_1)}\right](-2\sqrt{2})^N\bar\phi^{N-1}\times \\ &\times\sum_{\{\eta_2,\ldots,\eta_N\}}\frac{1}{(2m_1)!}\cdots\frac{1}{(2m_{N-1})!}\sum_{\sigma_2}\cdots\sum_{\sigma_N}\times \\ &\times\left[\prod_{\{n|\eta_n=+1\}}\alpha_n\right]\left[\prod_{\{n|\eta_n=0\}}\beta_n\epsilon^{j_{2n-1}j_{\sigma_n(k_n)}}\right]\left[\prod_{\{n|\eta_n=-1\}}\gamma_n\epsilon^{j_{2n}j_{\sigma_n(l_n)}}\epsilon^{j_{2n-1}j_{\sigma_n(k_n)}}\right]\times \\ ={}& \frac{1}{\sqrt{2}}C_s + \frac{2}{3}C_2\bar\phi + \frac{2\sqrt{2}}{3}C_3\bar\phi^2 + \mathcal{O}(C^4)\,, \end{aligned} \quad (4.40)$$

where a series $\{\eta_2, \eta_3, \ldots, \eta_N\}$ is the similar one defined by (4.25) but the condition (4.26) replaced with

$$m_n \geq 1 \qquad \text{for } {}^\forall n \leq N-1\,, \qquad m_N = 0\,, \quad (4.41)$$

and

$$\mathrm{Tr}\left[C_{(j_1j_2)}\cdot C_{(j_3j_4)}\cdots C_{(j_{2N-1}j_{2N})}\right] \equiv \varepsilon_{\alpha\beta}\left(C_{(j_1j_2)}\cdot C_{(j_3j_4)}\cdots C_{(j_{2N-1}j_{2N})}\right)^{\beta\alpha}, \quad (4.42)$$

$$C_2 \equiv \mathrm{Tr}\left[C_{(ij)}\cdot C_{(kl)}\right]\epsilon^{jk}\epsilon^{li}\,, \quad (4.43)$$

$$C_3 \equiv \mathrm{Tr}\left[C_{(ij)}\cdot C_{(kl)}\cdot C_{(mn)}\right]\epsilon^{jk}\epsilon^{lm}\epsilon^{ni}\,. \quad (4.44)$$

Note that, the second term of eq. (4.39) may be naively written with a factor

$$\left(C_{(j_{2N}j_{2N-1})}\cdot C_{(j_{2N-2}j_{2N-3})}\cdots C_{(j_2j_1)}\right)^{\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ (-2\sqrt{2}\bar\phi)^N\,,$$

in accordance with the general form of $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$ given in (4.23), but this factor can be simplified as written in (4.39) and (4.40) though it is possible only in the sum, as explained below. Let us write a specific term in the sum corresponding to a given series $\{\eta_2, \ldots, \eta_{N-1}, \eta_N\}$ as

$$c \times \mathcal{C}_{(N)}^{\alpha\beta}(\sigma_\mu \bar{\sigma}^\nu \varepsilon)_{\alpha\beta}\partial_\nu\lambda\ (-2\sqrt{2}\bar{\phi})^N\,, \tag{4.45}$$

where $\mathcal{C}_{(N)}^{\alpha\beta}$ is defined below eq. (4.23), and c is a constant. We can easily see that another series given by $\{-\eta_{N-1}, -\eta_{N-2}, \ldots, -\eta_2, \eta_N\}$ will produce a sign reversed, the spinor indices of $\mathcal{C}$ exchanged contribution:

$$-c \times \mathcal{C}_{(N)}^{\beta\alpha}(\sigma_\mu \bar{\sigma}^\nu \varepsilon)_{\alpha\beta}\partial_\nu\lambda\ (-2\sqrt{2}\bar{\phi})^N\,. \tag{4.46}$$

For example, for $N = 5$, there is a contribution corresponding to a series $\{0, +1, -1, -1\}$ which can be written as

$$+\frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{3}\frac{3}{20}\frac{1}{3}\mathcal{C}_{(5)}^{\alpha\beta}(\sigma_\mu \bar{\sigma}^\nu \varepsilon)_{\alpha\beta}\partial_\nu\lambda\left(-2\sqrt{2}\bar{\phi}\right)^5, \tag{4.47}$$

where

$$\mathcal{C}_{(5)}^{\alpha\beta} \equiv \epsilon^{j_{10}j_6}\epsilon^{j_9j_5}\epsilon^{j_8j_4}\epsilon^{j_7j_2}\epsilon^{j_3j_1}\left(C_{(j_{10}j_9)}\cdot C_{(j_8j_7)}\cdot C_{(j_6j_5)}\cdot C_{(j_4j_3)}\cdot C_{(j_2j_1)}\right)^{\alpha\beta}. \tag{4.48}$$

Note that here we have suppressed the possible permutations of the indices j_2, j_4, j_5 and j_6 by representing them with the one contribution (whose each contraction is taken with the rightmost possible SU(2) index of the factor $(C\cdots C)^{\alpha\beta}$). On the other hand, $\{+1, -1, 0, -1\}$ gives

$$+\frac{1}{4}\frac{1}{3}\frac{3}{20}\left(-\frac{1}{2}\right)\frac{1}{3}\mathcal{C'}_{(5)}^{\alpha\beta}\left(\sigma_\mu \bar{\sigma}^\nu \varepsilon\right)_{\alpha\beta}\partial_\nu\lambda\left(-2\sqrt{2}\bar{\phi}\right)^5, \tag{4.49}$$

where

$$\begin{aligned}
\mathcal{C'}_{(5)}^{\alpha\beta} &\equiv \epsilon^{j_{10}j_8}\epsilon^{j_9j_4}\epsilon^{j_7j_3}\epsilon^{j_6j_2}\epsilon^{j_5j_1}\left(C_{(j_{10}j_9)}\cdot C_{(j_8j_7)}\cdot C_{(j_6j_5)}\cdot C_{(j_4j_3)}\cdot C_{(j_2j_1)}\right)^{\alpha\beta}\\
&= \epsilon^{j_1j_4}\epsilon^{j_2j_7}\epsilon^{j_4j_8}\epsilon^{j_5j_9}\epsilon^{j_6j_{10}}\left(C_{(j_1j_2)}\cdot C_{(j_3j_4)}\cdot C_{(j_5j_6)}\cdot C_{(j_7j_8)}\cdot C_{(j_9j_{10})}\cdot\right)^{\alpha\beta}\\
&= (-1)^5\epsilon^{j_{10}j_8}\epsilon^{j_9j_4}\epsilon^{j_7j_3}\epsilon^{j_6j_2}\epsilon^{j_5j_1}(-1)^4\left(C_{(j_{10}j_9)}\cdot C_{(j_8j_7)}\cdot C_{(j_6j_5)}\cdot C_{(j_4j_3)}\cdot C_{(j_2j_1)}\right)^{\beta\alpha}\\
&= -\mathcal{C}_{(5)}^{\beta\alpha}\,.
\end{aligned} \tag{4.50}$$

Here we have renamed the dummy indices in the second line and used a property of the quantity (4.24),

$$\left(C_{(j_1j_2)}\cdot C_{(j_3j_4)}\cdots C_{(j_{2N-1}j_{2N})}\right)^{\alpha\beta} = (-1)^{N-1}\left(C_{(j_{2N}j_{2N-1})}\cdot C_{(j_{2N-3}j_{2N-4})}\cdots C_{(j_2j_1)}\right)^{\beta\alpha}. \tag{4.51}$$

in the third line. Therefore, taken into account the summation over the series, $(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}$ above can be reduced to $-\eta_\mu{}^\nu\varepsilon_{\alpha\beta}$:

$$c \times \mathrm{Tr}[\mathcal{C}_{(N)}](-2\sqrt{2}\ \bar{\phi})^N\partial_\mu\lambda\,. \tag{4.52}$$

By a direct calculation with the use of eq. (4.19), together with $(\delta^*_{\lambda_C} A_\mu)_{(0)}$ and $(\delta^*_{\lambda_C} A_\mu)_{(1)}$ given in eq. (4.9) and (4.10) respectively, we find that $\delta^*_{\lambda_C} A_\mu$ is given by

$$\delta^*_{\lambda_C} A_\mu = -\partial_\mu\lambda - \frac{1}{\sqrt{2}} C_s \partial_\mu\lambda\ \bar{\phi} - \frac{2}{3} C_2 \partial_\mu\lambda\ \bar{\phi}^2 - \frac{2\sqrt{2}}{3} C_3 \partial_\mu\lambda\ \bar{\phi}^3 + \mathcal{O}(C^4)\,, \tag{4.53}$$

and we can check that this is actually coincide with the result given by the formula (4.39).

4.2 The deformed gauge transformations of the other fields

Next we consider the $(\bar{\theta}^+)^2$-component of $\delta^*_{\lambda_C} V^{++}_{WZ}$ to determine $\delta^*_{\lambda_C}\phi$ and $\lambda^{(-2)}$. The equation to solve is read from (4.4) and (4.5):

$$\begin{aligned}\sqrt{2}i\delta^*_{\lambda_C}\phi &= iC_s\partial_\mu\lambda\ A^\mu - 2iC^{(+-)\alpha\beta}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ A^\mu + \\ &\quad + C^{++\alpha\beta}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\lambda^{(-2)}_\nu A^\mu - \partial^{++}\lambda^{(-2)}\,,\end{aligned} \tag{4.54}$$

where $\delta^*_{\lambda_C}\phi$ is identified with the lowest order term in the harmonic expansion of the left hand side (i.e. the U(1) charge 0, irreps of isospin 0 part). Expanding $\delta^*_{\lambda_C}\phi$ and $\lambda^{(-2)}$ as

$$\delta^*_{\lambda_C}\phi \equiv \sum_{N=0}^{\infty}(\delta^*_{\lambda_C}\phi)_{(N)}\,, \tag{4.55}$$

where $(\delta^*_{\lambda_C}\phi)_{(N)}$ is an $\mathcal{O}(C^N)$-quantity, eq. (4.54) gives the following equations to determine $\delta^*_{\lambda_C}\phi$:

$$(\delta^*_{\lambda_C}\phi)_{(1)} = \frac{1}{\sqrt{2}} C_s\partial_\mu\lambda\ A^\mu\,, \tag{4.56}$$

$$(\delta^*_{\lambda_C}\phi)_{(N)} = \frac{i}{3\sqrt{2}}\epsilon^{ik}\epsilon^{jl}C^{\alpha\beta}_{(ij)}(\lambda^{(-2)}_\nu{}_{(N-1)})_{(kl)}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}A^\mu \qquad (N\geq 2)\,, \tag{4.57}$$

where we have used the reduction formula (4.14). Comparing this equation to eq. (4.19), we find that $\delta^*_{\lambda_C}\phi$ can be obtained by replacing one of $\bar{\phi}$ in $\delta^*_{\lambda_C} A_\mu$ with $-A^\mu$: indeed, we have

$$\delta^*_{\lambda_C}\phi = \frac{1}{\sqrt{2}} C_s\partial_\mu\lambda\ A^\mu + \frac{2}{3} C_2\partial_\mu\lambda\ A^\mu\bar{\phi} + \frac{2\sqrt{2}}{3} C_3\partial_\mu\lambda\ A^\mu\bar{\phi}^2 + \mathcal{O}(C^4)\,. \tag{4.58}$$

Thus the full form of $\delta^*_{\lambda_C}\phi$ can be formulated as the one quite similar to eq. (4.39):

$$\delta^*_{\lambda_C}\phi = f(\bar{\phi})\partial_\mu\lambda\ A^\mu\,, \tag{4.59}$$

where $f(\bar{\phi})$ is defined in eq. (4.40).

Eq. (4.54) also gives the equations to determine $\lambda^{(-2)}$:

$$\partial^{++}\lambda^{(-2)}{}_{(1)} = -2iC^{(+-)\alpha\beta}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ A^\mu\,, \tag{4.60}$$

$$\partial^{++}\lambda^{(-2)}{}_{(N)} = C^{++\alpha\beta}(\sigma_\mu\bar{\sigma}^\nu\varepsilon)_{\alpha\beta}\lambda^{(-2)}_\nu{}_{(N-1)}A^\mu + \sqrt{2}i(\delta^*_{\lambda_C}\phi)_{(N)} \qquad (N\geq 2)\,. \tag{4.61}$$

The first equation is easily solved to give $\lambda^{(-2)}{}_{(1)}$, and the second equation can be used to determine $\lambda^{(-2)}{}_{(N)}$ from $\lambda^{(-2)}_\mu{}_{(N-1)}$ and $(\delta^*_{\lambda_C}\phi)_{(N)}$ given above. Comparing these equations to eqs. (4.11) and (4.12), it is found that $\lambda^{(-2)}$ can be obtained by replacing one of ϕ in $\lambda_\mu^{(-2)}$ with $-A^\mu/\sqrt{2}$. Of course, with the harmonic expansion of $\lambda^{(-2)}$, we can construct the recursion relation in terms of the coefficient functions in the harmonic expansion, which is similar to eq. (4.15). The expansion is

$$\lambda^{(-2)} \equiv \sum_{N=1}^{\infty} \lambda^{(-2)}{}_{(N)}\,, \tag{4.62}$$

$$\lambda^{(-2)}{}_{(N)} = \sum_{n=0}^{\infty} (\lambda^{(-2)}{}_{(N)})_{(i_1\ldots i_n k_1 \ldots k_{n+2})} u^{+i_1}\ldots u^{+i_n} u^{-k_1}\ldots u^{-k_{n+2}}\,, \tag{4.63}$$

where $\lambda^{(-2)}{}_{(N)}$ is an $\mathcal{O}(C^N)$-quantity. Note again that the C-independent part of $\lambda^{(-2)}$ is absent, because of its absence in the $C\to 0$ limit. Then $\lambda^{(-2)}$ is determined as

$$\begin{aligned}\lambda^{(-2)} &= -iC^{--\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ A^\mu + 2\sqrt{2}i\left(C_{(ij)}\cdot C_{(kl)}\right)^{\alpha\beta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\nu\lambda\ A^\mu\bar\phi\ \times\\ &\quad\times\left(\frac{1}{3}u^{+(i}u^{-j}u^{-k}u^{-l)} - \frac{1}{2}\epsilon^{jl}u^{-(i}u^{-k)}\right) + \mathcal{O}(C^3)\,.\end{aligned} \tag{4.64}$$

For $\delta^*_{\lambda_C}\psi^i$, the equation to solve is read from (4.4) and (4.5):

$$\begin{aligned}&-2C_s\partial_\mu\lambda\ (\sigma^\mu\bar\psi^-)_\alpha - 4C^{(+-)\beta\gamma}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\beta\gamma}\partial_\nu\lambda\ (\sigma^\mu\bar\psi^-)_\alpha -\\ &-2iC^{++\beta\gamma}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\beta\gamma}\lambda_\nu^{(-2)}(\sigma^\mu\bar\psi^-)_\alpha - \partial^{++}\lambda_\alpha^{(-3)} - 2\sqrt{2}(\varepsilon C^{++}\lambda^{(-3)})_\alpha\bar\phi = 4\delta^*_{\lambda_C}\psi^-_\alpha\,,\end{aligned} \tag{4.65}$$

where $\delta^*_{\lambda_C}\psi^-$ is identified with the lowest order term in the harmonic expansion of the left hand side (i.e. the U(1) charge -1, irreps of isospin 1/2 part). Expand $\delta^*_{\lambda_C}\psi^-$ and $\lambda_\alpha^{(-3)}$ as

$$\delta^*_{\lambda_C}\psi^- \equiv \sum_{N=0}^{\infty}(\delta^*_{\lambda_C}\psi^-)_{(N)}\,, \tag{4.66}$$

$$\lambda_\alpha^{(-3)} \equiv \sum_{N=1}^{\infty}\lambda_\alpha^{(-3)}{}_{(N)}\,, \tag{4.67}$$

$$\lambda_\alpha^{(-3)}{}_{(N)} = \sum_{n=0}^{\infty}(\lambda_\alpha^{(-3)}{}_{(N)})_{(i_1\ldots i_n k_1\ldots k_{n+3})}u^{+i_1}\ldots u^{+i_n}u^{-k_1}\ldots u^{-k_{n+3}}\,. \tag{4.68}$$

where $(\delta^*_{\lambda_C}\psi^-)_{(N)}$ and $\lambda_\alpha^{(-3)}{}_{(N)}$ are $\mathcal{O}(C^N)$-quantities. Then eq. (4.65) gives the following equations:

$$(\delta^*_{\lambda_C}\psi^-_\alpha)_{(1)} = -\frac{1}{2}C_s\partial_\mu\lambda\ (\sigma^\mu\bar\psi^-)_\alpha - \frac{2}{3}C^{\gamma\delta}_{(ik)}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\gamma\delta}\partial_\nu\lambda(\sigma^\mu\bar\psi^k)_\alpha u^{-i}\,, \tag{4.69}$$

$$\partial^{++}\lambda_\alpha^{(-3)}{}_{(1)} = -4C^{\beta\gamma}_{(ij)}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\beta\gamma}\partial_\nu\lambda\ (\sigma^\mu\bar\psi_k)_\alpha u^{+(i}u^{-j}u^{-k)}\,, \tag{4.70}$$

$$\begin{aligned}\partial^{++}\lambda_\alpha^{(-3)}{}_{(N)} &= -2iC^{++\gamma\delta}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\gamma\delta}\lambda_\nu^{(-2)}{}_{(N-1)}(\sigma^\mu\bar\psi^-)_\alpha -\\ &\quad - 2\sqrt{2}(\varepsilon C^{++}\lambda^{(-3)}{}_{(N-1)})_\alpha\bar\phi - 4(\delta^*_{\lambda_C}\psi^-_\alpha)_{(N)} \qquad (N\geq 2)\,.\end{aligned} \tag{4.71}$$

The second and the third equation will lead to $\lambda_\alpha^{(-3)}$ as

$$\begin{aligned}\lambda_\alpha^{(-3)} &= +\frac{8}{3}(\varepsilon C^{--}\sigma^\mu\bar\psi^-)_\alpha\partial_\mu\lambda + \frac{8\sqrt{2}}{3}\,\mathrm{Tr}\left[C_{(mn)}\cdot C_{(ij)}\right]\partial_\mu\lambda(\sigma^\mu\bar\psi_k)_\alpha\bar\phi\ \times\\ &\quad\times\left(u^{-(m}u^{-n}u^{-i}u^{-j}u^{-k)} - \frac{2}{5}\epsilon^{mk}u^{+(n}u^{-i}u^{-j)}\right) + \mathcal{O}(C^3)\,.\end{aligned} \tag{4.72}$$

Noting again the reduction formula (4.14), we find from the third equation that $(\delta^*_{\lambda_C}\psi^-)_{(N)}$ is given by

$$\begin{aligned}(\delta^*_{\lambda_C}\psi^-_\alpha)_{(N)} \;=\; &-\frac{i}{6}\epsilon^{jn}C^{\gamma\delta}_{(ij)}(\lambda^{(-2)}_\nu{}_{(N-1)})_{(mn)}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\gamma\delta}(\sigma^\mu\bar\psi_k)_\alpha(\epsilon^{k(m}u^{-i)}-\epsilon^{im}u^{-k})+\\ &+\frac{i}{20}\epsilon^{ip}\epsilon^{jq}C^{\gamma\delta}_{(ij)}(\lambda^{(-2)}_\nu{}_{(N-1)})_{(mnpq)}(\sigma_\mu\bar\sigma^\nu\varepsilon)_{\gamma\delta}(\sigma^\mu\bar\psi_k)_\alpha\epsilon^{kn}u^{-m}-\\ &-\frac{\sqrt2}{4}\epsilon^{il}\epsilon^{jm}\left(\varepsilon C_{(ij)}\cdot(\lambda^{(-3)}{}_{(N-1)})_{(klm)}\right)_\alpha\bar\phi\;u^{-k}\qquad(N\geq 2)\,.\end{aligned}\tag{4.73}$$

Although we do not write down general formula for $\delta^*_{\lambda_C}\psi^-$ in the present work, the lower order terms are given by

$$\begin{aligned}\delta^*_{\lambda_C}\psi^-_\alpha \;=\; &-\frac{1}{2}C_s(\sigma^\mu\bar\psi^-)_\alpha\,\partial_\mu\lambda-\frac{2}{3}(\varepsilon C_{(ij)}\sigma^\mu\bar\psi^j)_\alpha\,\partial_\mu\lambda\,u^{-i}-\\ &-\frac{5\sqrt2}{3}(\varepsilon C_{(kl)}\;\varepsilon C_{(ij)}\sigma^\mu\bar\psi_k)_\alpha\,\bar\phi\,\partial_\mu\lambda\left(\frac{2}{3}\epsilon^{mi}\epsilon^{n(j}u^{-k)}+\frac{1}{3}\epsilon^{mk}\epsilon^{ni}u^{-j}\right)+\\ &+(\varepsilon C_{(ij)}\;\varepsilon C_{(kl)}\;\varepsilon C_{(mn)}\sigma^\mu\bar\psi_p)_\alpha\,\bar\phi^2\,\partial_\mu\lambda\,\times\\ &\times\left\{\frac{1}{5}\epsilon^{mm'}\epsilon^{nn'}\delta_{m'}{}^{(i}\delta_{n'}{}^{j}\epsilon^{pk}u^{-l)}+\right.\\ &\quad+\frac{1}{15}\epsilon^{mm'}\epsilon^{nn'}\left(\delta_{m'}{}^{(i}\delta_{n'}{}^{j)}\epsilon^{p(k}u^{-l)}+\delta_{m'}{}^{(k}\delta_{n'}{}^{l)}\epsilon^{p(i}u^{-j)}\right)+\\ &\quad\left.+2\epsilon^{mm'}\epsilon^{pp'}\epsilon^{ki}\delta_{m'}{}^{(l}\delta_{p'}{}^{n}u^{-j)}+\frac{32}{9}\epsilon^{kk'}\epsilon^{mm'}\epsilon^{nn'}\delta_{k'}{}^{(p}\delta_{m'}{}^{i}\delta_{n'}{}^{j}u^{-l)}\right\}+\\ &+\mathcal{O}(C^4).\end{aligned}\tag{4.74}$$

For $\delta^*_{\lambda_C}D^{ij}$, from (4.4) and (4.5), the equation to solve is given by

$$4\sqrt2C^{--\mu\nu}\partial_\mu\lambda\;\partial_\nu\bar\phi-i\partial^\mu\lambda^{(-2)}_\mu-\sqrt2iC^{+-\alpha\beta}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\nu(\lambda^{(-2)}_\mu\bar\phi)-\partial^{++}\lambda^{(-4)}=3\delta^*_{\lambda_C}D^{--}\,.\tag{4.75}$$

where $\delta^*_{\lambda_C}D^{--}$ is identified with the lowest order term in the harmonic expansion of the left hand side (i.e. the U(1) charge -2, irreps of isospin 1 part). Expanding $\delta^*_{\lambda_C}D^{--}$ as

$$\delta^*_{\lambda_C}D^{--}\equiv\sum_{N=0}^{\infty}(\delta^*_{\lambda_C}D^{--})_{(N)}\,,\tag{4.76}$$

where $(\delta^*_{\lambda_C}D^{--})_{(N)}$ is an $\mathcal{O}(C^N)$-quantity. then eq. (4.54) gives the following equations to determine $\delta^*_{\lambda_C}D^{--}$:

$$(\delta^*_{\lambda_C}D^{--})_{(1)}\;=\;2\sqrt2C^{--\mu\nu}\partial_\mu\lambda\;\partial_\nu\bar\phi,\tag{4.77}$$

$$\begin{aligned}(\delta^*_{\lambda_C}D^{--})_{(N)}\;=\;&\Bigg[-\frac{i}{3}\partial^\mu(\lambda^{(-2)}_\mu{}_{(N)})_{(ij)}-\frac{\sqrt2}{6}iC_s\partial^\mu\left((\lambda^{(-2)}_\mu{}_{(N-1)})_{(ij)}\bar\phi\right)-\\ &-\frac{i}{\sqrt2}\epsilon^{kl}C^{\alpha\beta}_{(ik)}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\mu\left((\lambda^{(-2)}_\nu{}_{(N-1)})_{(jl)}\bar\phi\right)-\\ &-\frac{3\sqrt2}{20}i\epsilon^{km}\epsilon^{ln}C^{\alpha\beta}_{(kl)}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\mu\times\\ &\times\left((\lambda^{(-2)}_\nu{}_{(N-1)})_{(ijmn)}\bar\phi\right)\Bigg]u^{-i}u^{-j}\qquad(N\geq 2)\,.\end{aligned}\tag{4.78}$$

Here we have used another reduction formula similar to eq. (4.14):

$$\begin{aligned}
&C^{(+-)\alpha\beta}(\lambda_\mu^{(-2)}{}_{(N)})_{(j_{2m}j_{2m-1}\dots j_1)}\overbrace{u^{+(j_{2m}}\dots u^{+j_{m+2}}}^{m-1}\overbrace{u^{-j_{m+1}}\dots u^{-j_1)}}^{m+1} = \\
&\quad = C^{\alpha\beta}_{(j_{2m+2}j_{2m+1})}(\lambda_\mu^{(-2)}{}_{(N)})_{(j_{2m}j_{2m-1}\dots j_1)} \times \\
&\quad \times \Bigg(\overbrace{u^{+(j_{2m+2}}u^{+j_{2m+1}}u^{+j_{2m}}\dots u^{+j_{m+3}}}^{m}\overbrace{u^{-j_{m+2}}\dots u^{-j_1)}}^{m+2} - \\
&\qquad - \frac{1}{m+1}\epsilon^{j_{2m+2}j_1}\overbrace{u^{+(j_{2m+1}}u^{+j_{2m}}\dots u^{+j_{m+3}}}^{m-1}\overbrace{u^{-j_{m+2}}\dots u^{-j_2)}}^{m+1} - \\
&\qquad - \frac{(m-1)(m+1)}{2m(2m+1)}\epsilon^{j_{2m+2}j_2}\epsilon^{j_{2m+1}j_1}\overbrace{u^{+(j_{2m}}\dots u^{+j_{m+3}}}^{m-2}\overbrace{u^{-j_{m+2}}\dots u^{-j_3)}}^{m}\Bigg) . \qquad (4.79)
\end{aligned}$$

Note that for $m = 1$ the last term vanishes.

With the use of eq. (4.15), the last term in eq. (4.78) can be rewritten with $(\lambda_\mu^{(-2)}{}_{(N)})_{(ij)}$ and $(\lambda_\mu^{(-2)}{}_{(N-1)})_{(ij)}$, thus the equation becomes

$$\begin{aligned}
(\delta^*_{\lambda_C}D^{--})_{(N)} &= -\frac{\sqrt{2}}{6}iC_s\partial^\mu\left((\lambda_\mu^{(-2)}{}_{(N-1)})_{(ij)}\bar\phi\right)u^{-i}u^{-j} - \\
&\quad -\frac{i}{\sqrt{2}}\epsilon^{kl}C^{\alpha\beta}_{(ik)}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\mu\left((\lambda_\nu^{(-2)}{}_{(N-1)})_{(jl)}\bar\phi\right)u^{-i}u^{-j} . \qquad (4.80)
\end{aligned}$$

Recall that $\lambda_\mu^{(-2)}{}_{(N)}$ has been already explicitly given in eq. (4.28). Thus, substituting the previous result, a full form of $\delta^*_{\lambda_C}D^{--}$ is given by

$$\begin{aligned}
\delta^*_{\lambda_C}D^{--} &= 2\sqrt{2}C^{--\mu\nu}\partial_\mu\lambda\partial_\nu\bar\phi + \frac{\sqrt{2}}{6}iC_sh^{--}{}_\mu{}^\nu(\bar\phi)\partial^\mu(\partial_\nu\lambda\ \bar\phi) + \\
&\quad +\frac{i}{\sqrt{2}}\epsilon^{kl}C^{\alpha\beta}_{(ik)}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}h_{(jl)\nu}{}^\rho(\bar\phi)\partial_\mu(\partial_\rho\lambda\ \bar\phi)\ u^{-i}u^{-j} \\
&= 2\sqrt{2}C^{--\mu\nu}\partial_\mu\lambda\partial_\nu\bar\phi - \frac{2}{3}C_sC^{--\mu\nu}\bar\phi\partial_\mu(\partial_\nu\lambda\ \bar\phi) + \\
&\quad +8\epsilon^{kl}\left(C_{(ik)}\cdot C_{(jl)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar\phi\ \partial_\mu(\partial_\nu\lambda\ \bar\phi)\ u^{-i}u^{-j} + \\
&\quad +\frac{2\sqrt{2}}{3}C_s\epsilon^{mn}\left(C_{(im)}\cdot C_{(jn)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar\phi^2\partial_\mu(\partial_\nu\lambda\ \bar\phi)u^{-i}u^{-j} + \\
&\quad +4\sqrt{2}\epsilon^{mn}\left(C_{(ik)}\cdot C_{(jm)}\cdot C_{(ln)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar\phi^2\partial_\mu(\partial_\nu\lambda\ \bar\phi)u^{-i}\epsilon^{k(l}u^{-j)} + \\
&\quad +\mathcal{O}(C^4) . \qquad (4.81)
\end{aligned}$$

Expanding $\lambda^{(-4)}$ as

$$\lambda^{(-4)} \equiv \sum_{N=1}^{\infty}\lambda^{(-4)}{}_{(N)} , \qquad (4.82)$$

$$\lambda^{(-4)}{}_{(N)} = \sum_{n=0}^{\infty}(\lambda^{(-4)}{}_{(N)})_{(i_1\dots i_nk_1\dots k_{n+4})}u^{+i_1}\dots u^{+i_n}u^{-k_1}\dots u^{-k_{n+4}} , \qquad (4.83)$$

where $\lambda^{(-4)}{}_{(N)}$ is an $\mathcal{O}(C^N)$-quantity, eq. (4.75) also gives the equations to determine $\lambda^{(-4)}{}_{(N)}$:

$$\partial^{++}\lambda^{(-4)}{}_{(N)} = -i\partial^{\mu}\lambda_{\mu}^{(-2)}{}_{(N)} - \sqrt{2}iC^{+-\alpha\beta}(\sigma^{\mu}\bar{\sigma}^{\nu}\varepsilon)_{\alpha\beta}\partial_{\nu}\left(\lambda_{\mu}^{(-2)}{}_{(N-1)}\bar{\phi}\right) - 3(\delta^{*}_{\lambda_C}D^{--})_{(N)} \,. \tag{4.84}$$

In principle, we can calculate $\lambda^{(-4)}$ order by order in C with the use of the previous results. The lower terms are given by

$$\lambda^{(-4)} = \frac{1}{3}\,\mathrm{Tr}\left[C^{--}\cdot C^{--}\right]\partial_{\mu}(\bar{\phi}^2\,\partial^{\mu}\lambda) + \mathcal{O}(C^3)\,. \tag{4.85}$$

We will not, however, write down the full form here. What is important is that it is possible to completely gauge away the left hand side of eq. (4.75) with $\lambda^{(-4)}$, after the subtraction of $\delta^{*}_{\lambda_C}D^{--}$ given above.

To summarize, the gauge transformations at the order $\mathcal{O}(C^3)$ are given by

$$\begin{aligned}
\delta^{*}_{\lambda_C}A_{\mu} &= -\partial_{\mu}\lambda - \frac{1}{\sqrt{2}}C_s\partial_{\mu}\lambda\;\bar{\phi} - \frac{2}{3}C_2\partial_{\mu}\lambda\;\bar{\phi}^2 - \frac{2\sqrt{2}}{3}C_3\partial_{\mu}\lambda\;\bar{\phi}^3 + \mathcal{O}(C^4)\,,\\
\delta^{*}_{\lambda_C}\phi &= \frac{1}{\sqrt{2}}C_s\partial_{\mu}\lambda\;A^{\mu} + \frac{2}{3}C_2\partial_{\mu}\lambda\;A^{\mu}\bar{\phi} + \frac{2\sqrt{2}}{3}C_3\partial_{\mu}\lambda\;A^{\mu}\bar{\phi}^2 + \mathcal{O}(C^4)\,,\\
\delta^{*}_{\lambda_C}\psi^{-}_{\alpha} &= -\frac{1}{2}C_s(\sigma^{\mu}\bar{\psi}^{-})_{\alpha}\,\partial_{\mu}\lambda - \frac{2}{3}(\varepsilon C_{(ij)}\sigma^{\mu}\bar{\psi}^{j})_{\alpha}\,\partial_{\mu}\lambda\,u^{-i} - \\
&\quad - \frac{5\sqrt{2}}{3}(\varepsilon C_{(mn)}\;\varepsilon C_{(ij)}\sigma^{\mu}\bar{\psi}_{k})_{\alpha}\,\bar{\phi}\,\partial_{\mu}\lambda\left(\frac{2}{3}\epsilon^{mi}\epsilon^{n(j}u^{-k)} + \frac{1}{3}\epsilon^{mk}\epsilon^{ni}u^{-j}\right) + \\
&\quad + (\varepsilon C_{(ij)}\;\varepsilon C_{(kl)}\;\varepsilon C_{(mn)}\sigma^{\mu}\bar{\psi}_{p})_{\alpha}\,\bar{\phi}^2\,\partial_{\mu}\lambda\;\times\\
&\quad \times\Bigg\{\frac{1}{5}\epsilon^{mm'}\epsilon^{nn'}\delta_{m'}{}^{(i}\delta_{n'}{}^{j}\epsilon^{pk}u^{-l)} + \\
&\qquad + \frac{1}{15}\epsilon^{mm'}\epsilon^{nn'}\left(\delta_{m'}{}^{(i}\delta_{n'}{}^{j)}\epsilon^{p(k}u^{-l)} + \delta_{m'}{}^{(k}\delta_{n'}{}^{l)}\epsilon^{p(i}u^{-j)}\right) + \\
&\qquad + 2\epsilon^{mm'}\epsilon^{pp'}\epsilon^{ki}\delta_{m'}{}^{(l}\delta_{p'}{}^{n}u^{-j)} + \frac{32}{9}\epsilon^{kk'}\epsilon^{mm'}\epsilon^{nn'}\delta_{k'}{}^{(p}\delta_{m'}{}^{i}\delta_{n'}{}^{j}u^{-l)}\Bigg\} + \\
&\quad + \mathcal{O}(C^4)\,,\\
\delta^{*}_{\lambda_C}D^{--} &= 2\sqrt{2}C^{--\mu\nu}\partial_{\mu}\lambda\partial_{\nu}\bar{\phi} - \frac{2}{3}C_sC^{--\mu\nu}\bar{\phi}\partial_{\mu}(\partial_{\nu}\lambda\;\bar{\phi}) + \\
&\quad + 8\epsilon^{kl}\left(C_{(ik)}\cdot C_{(jl)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar{\phi}\;\partial_{\mu}(\partial_{\nu}\lambda\;\bar{\phi})\;u^{-i}u^{-j} + \\
&\quad + \frac{2\sqrt{2}}{3}C_s\epsilon^{mn}\left(C_{(im)}\cdot C_{(jn)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar{\phi}^2\partial_{\mu}(\partial_{\nu}\lambda\;\bar{\phi})u^{-i}u^{-j} + \\
&\quad + 4\sqrt{2}\epsilon^{mn}\left(C_{(ik)}\cdot C_{(jm)}\cdot C_{(ln)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar{\phi}^2\partial_{\mu}(\partial_{\nu}\lambda\;\bar{\phi})u^{-i}\epsilon^{k(l}u^{-j)} + \\
&\quad + \mathcal{O}(C^4)\,,\\
\delta^{*}_{\lambda_C}(\text{others}) &= 0\,.
\end{aligned} \tag{4.86}$$

One can confirm that the terms at the order $\mathcal{O}(C^1)$ in the action $S_{*,2} + S_{*,3}$ are invariant under these transformations.

4.3 Field redefinitions and the action

We can redefine (reparameterize) the component fields in the gauge superfield V_{WZ}^{++} in order to make the gauge transformation become a convenient form. There are possibly many choices to achieve this. Here we will absorb all of the C-dependent terms emerging in the deformed gauge transformation by the reparameterization procedure so as to make the resulted component gauge transformation laws canonical (i.e. the ordinary abelian gauge transformations). Presumably we could reparameterize the coefficients so as not to fully extinguish the C-dependent terms from the resulted component gauge transformation laws. Then the procedure to absorb terms emerging above into the definition of the coefficient functions in the gauge superfield could be a very difficult task, because we have to find a parameterization which gives the deformed gauge transformation of the gauge superfield under the resulted partially deformed gauge transformation laws.

In order to make the component gauge transformation canonical, the following field redefinitions are sufficient:

$$A_\mu \longrightarrow \left(1+f(\bar\phi)\bar\phi\right)A_\mu = A_\mu + \frac{1}{\sqrt{2}}C_sA_\mu\bar\phi + \frac{2}{3}C_2A_\mu\bar\phi^2 + \mathcal{O}(C^3)\,, \tag{4.87}$$

$$\begin{aligned}\phi \longrightarrow \phi - \frac{1}{2}f(\bar\phi)\Big(1+f(\bar\phi)\bar\phi\Big)A_\mu A^\mu &= \phi - \frac{1}{2\sqrt{2}}C_sA_\mu A^\mu - \left(\frac{1}{4}C_s^2+\frac{1}{3}C_2\right)A_\mu A^\mu\bar\phi + \\ &+\mathcal{O}(C^3)\,,\end{aligned} \tag{4.88}$$

$$\begin{aligned}\psi^-_\alpha \longrightarrow\ &\psi^-_\alpha - \frac{1}{2}C_s(\sigma^\mu\bar\psi^-)_\alpha A_\mu - \frac{2}{3}(\varepsilon C_{(ij)}\sigma^\mu\bar\psi^j)_\alpha A_\mu u^{-i} - \\ &-\frac{5\sqrt{2}}{3}(\varepsilon C_{(mn)}\cdot C_{(ij)}\sigma^\mu\bar\psi_k)_\alpha\bar\phi A_\mu\left(\frac{2}{3}\epsilon^{mi}\epsilon^{n(j}u^{-k)}+\frac{1}{3}\epsilon^{mk}\epsilon^{ni}u^{-j}\right) + \\ &+\mathcal{O}(C^3),\end{aligned} \tag{4.89}$$

$$\begin{aligned}D^{--} \longrightarrow\ &D^{--} - 2\sqrt{2}C^{--\mu\nu}A_\mu\partial_\nu\bar\phi - \frac{\sqrt{2}}{6}iC_sh^{--}{}_\mu{}^\nu(\bar\phi)\partial^\mu(A_\nu\bar\phi) - \\ &-\frac{i}{\sqrt{2}}\epsilon^{kl}C^{\alpha\beta}_{(ik)}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}h_{(jl)\nu}{}^\rho(\bar\phi)\partial_\mu(A_\rho\bar\phi)\ u^{-i}u^{-j} = \\ &\quad = D^{--} - 2\sqrt{2}C^{--\mu\nu}A_\mu\partial_\nu\bar\phi + \frac{2}{3}C_sC^{--\mu\nu}\bar\phi\partial_\mu(A_\nu\bar\phi) - \\ &\qquad - 8\epsilon^{kl}\left(C_{(ik)}\cdot C_{(jl)}\right)^{\alpha\beta}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\bar\phi\ \partial_\mu(A_\nu\bar\phi)\ u^{-i}u^{-j} + \mathcal{O}(C^3)\,,\end{aligned} \tag{4.90}$$

where $f(\bar\phi)$ and $h^{--}{}_\mu{}^\nu(\bar\phi)$ are defined by eq. (4.40) and (4.29) respectively. There is no need to redefine the rest of the fields. We will denote the reparameterized gauge superfield as V_C^{++}. After this reparameterization, it becomes to hold that

$$\delta^*_{\tilde\lambda_C}V_C^{++} = \delta^0_\lambda V_C^{++}\,, \tag{4.91}$$

where $\tilde\lambda_C$ is given by replacing A_μ in λ_C with $\left(1+f(\bar\phi)\bar\phi\right)A_\mu$ and δ^0_λ denotes the ordinary component gauge transformations with gauge parameter $\lambda(x_A)$:

$$\delta^0_\lambda A_\mu \equiv -\partial_\mu\lambda\,, \qquad \delta^0_\lambda(\text{others}) \equiv 0\,. \tag{4.92}$$

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From eq. (4.91), we are allowed to consider that the gauge transformations of the component fields satisfy

$$\delta^*_{\tilde{\lambda}_C} A_\mu = \delta^0_\lambda A_\mu = -\partial_\mu \lambda\,, \qquad \delta^*_{\tilde{\lambda}_C}(\text{others}) = \delta^0_\lambda(\text{others}) = 0\,. \tag{4.93}$$

This means that the component gauge transformations can be thought of as the ordinary one.

After the field redefinition, the action in the WZ gauge will become to have manifestly gauge invariant form (in the sense of the ordinary component abelian gauge transformations), S_{*C}. Let us denote the contribution to this action from the n-th order in V_C^{++} as $S_{*C,n}$. It is seen that $S_{*,2}$ given in (3.20) changes to

$$\begin{aligned} S_{*C,2} = \int d^4x \Big[& -\frac{1}{4} F_{\mu\nu}(F^{\mu\nu} + \tilde{F}^{\mu\nu}) - \frac{1}{\sqrt{2}} C_s \partial_\mu (A_\nu \bar{\phi})(F^{\mu\nu} + \tilde{F}^{\mu\nu}) - \\ & - i\psi^i \sigma^\mu \partial_\mu \bar{\psi}_i + \frac{i}{2} C_s \varepsilon^{\alpha\beta} A_\mu (\sigma^\mu \bar{\psi}^k)_\alpha (\sigma^\nu \partial_\nu \bar{\psi}_k)_\beta - \\ & - \frac{2}{3} i C^{\alpha\beta}_{(ij)} A_\mu (\sigma^\mu \bar{\psi}^i)_\alpha (\sigma^\nu \partial_\nu \bar{\psi}^j)_\beta + \\ & + \phi \partial^2 \bar{\phi} - \frac{\sqrt{2}}{4} C_s A_\mu A^\mu \partial^2 \bar{\phi} \\ & + \frac{1}{4} D_{ij} D^{ij} - \sqrt{2} C^{\mu\nu}_{(ij)} D^{ij} A_\mu \partial_\nu \bar{\phi} + \mathcal{O}(C^2) \Big]\,. \end{aligned} \tag{4.94}$$

Thus we find

$$\begin{aligned} S_{*C,2} + S_{*C,3} = \int d^4x \Big[& -\frac{1}{4}(1 + \sqrt{2} C_s \bar{\phi}) F_{\mu\nu}(F^{\mu\nu} + \tilde{F}^{\mu\nu}) - \\ & - i\Big(1 - \frac{1}{\sqrt{2}} C_s \bar{\phi}\Big) \psi^i \sigma^\mu \partial_\mu \bar{\psi}_i + \frac{i}{\sqrt{2}} C_s \partial_\mu \bar{\phi} (\psi^i \sigma^\mu \bar{\psi}_i) - \\ & - 2\sqrt{2} i C^{\alpha\beta}_{(ij)} \psi^i_\alpha (\sigma^\mu \bar{\psi}^j)_\beta \partial_\mu \bar{\phi} - \frac{2\sqrt{2}}{3} i C^{\alpha\beta}_{(ij)} \psi^i_\alpha (\sigma^\mu \partial_\mu \bar{\psi}^j)_\beta \bar{\phi} + \\ & + \frac{i}{2} C_s \bar{\psi}^i \bar{\psi}^j D_{ij} - i C^{\mu\nu}_{(ij)} \bar{\psi}^i \bar{\psi}^j F_{\mu\nu} + \phi \partial^2 \bar{\phi} + \\ & + \frac{1}{4}(1 - \sqrt{2} C_s \bar{\phi}) D_{ij} D^{ij} + \frac{1}{\sqrt{2}} C^{\mu\nu}_{(ij)} D^{ij} F_{\mu\nu} \bar{\phi} + \mathcal{O}(C^2) \Big]\,. \end{aligned} \tag{4.95}$$

All the terms emerging here are manifestly gauge invariant as expected from the field redefinition procedure. We note that the kinetic terms of the fields show similar $\bar{\phi}$ dependence as in [10]. We expect that the full action can be written in terms of a function of $\bar{\phi}$, which is left for future study.

5. Conclusions and discussion

In this paper, we have studied $\mathcal{N} = 2$ supersymmetric U(1) gauge theory on the noncommutative harmonic superspace. We investigated the deformed gauge transformation which preserves the WZ gauge in detail. We also calculated the deformed lagrangian explicitly up to the third order in component fields. The general gauge transformation includes the arbitrary power of anti-holomorphic scalar fields, which does not appear in noncommutative $\mathcal{N}=1$ superspace. Generalization to $\mathcal{N}=2$ U(N) gauge theory is straightforward. Since the deformation is rather different from the $\mathcal{N}=1$ theory, it would be an interesting problem to study the perturbative and nonperturbative structure of deformed $\mathcal{N}=2$ theory.

In this paper we have studied the gauge symmetry but not supersymmetry. In order to study supersymmetry of the theory, one must act the supercharges Q^i_α on the gauge superfield V^{++}_{WZ}. This transformation does not preserve the WZ gauge. Therefore we need to make further gauge transformation to recover the WZ gauge. Because this gauge transformation is very complicated, we left this problem in a subsequent paper.

Another interesting problem is to study the deformed $\mathcal{N}=2$ theory from the D-brane viewpoint. In order to study this problem , it would be natural to investigate the hybrid formalism compactified on the $K3$ surface or four-dimensional torus. The theory on the D-brane provides the $\mathcal{N}=2$ supersymmetric Yang-Mills theory in four dimensions. Since the six-dimensional hybrid superstring includes the harmonic superspace structure [18], it is also interesting to clarify the relation between both harmonic superspace structures.

Acknowledgments

The works of T.A and A.O were supported in part by a 21st Century COE Program at TokyoTech "Nanometer-Scale Quantum Physics" by the Ministry of Education, Culture, Sports, Science and Technology.

A. Star product for analytic superfields

The $*$-product $A*B$ for analytic superfields A and B is given by

$$\begin{aligned}A*B =\ & AB-(-1)^{|A|}\frac{1}{2}Q^i_\alpha A C^{\alpha\beta}_{ij}Q^j_\beta B-\frac{1}{8}(Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}A)C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}(Q^{j_2}_{\beta_2}Q^{j_1}_{\beta_1})B+\\ &+\frac{1}{48}(-1)^{|A|}(Q^{i_3}_{\alpha_3}Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}A)C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}C^{\alpha_3\beta_3}_{i_3j_3}(Q^{j_3}_{\beta_3}Q^{j_2}_{\beta_2}Q^{j_1}_{\beta_1}B)+\\ &+\frac{1}{384}(Q^{i_4}_{\alpha_4}Q^{i_3}_{\alpha_3}Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}A)C^{\alpha_1\beta_1}_{i_1j_1}C^{\alpha_2\beta_2}_{i_2j_2}C^{\alpha_3\beta_3}_{i_3j_3}C^{\alpha_4\beta_4}_{i_4j_4}(Q^{j_4}_{\beta_4}Q^{j_3}_{\beta_3}Q^{j_2}_{\beta_2}Q^{j_1}_{\beta_1}B)\,. \end{aligned}\tag{A.1}$$

Here for an analytic superfield Φ of the form

$$\begin{aligned}\Phi =\ & \phi(x_A,u)+\theta^+\psi(x_A,u)+\bar\theta^+\bar\chi(x_A,u)+(\theta^+)^2M(x_A,u)+(\bar\theta^+)^2N(x_A,u)+ \\ &+\theta^+\sigma^m\bar\theta^+A_m(x_A,u)+(\bar\theta^+)^2\theta^+\lambda(x_A,u)+(\theta^+)^2\bar\theta^+\bar\kappa(x_A,u)+(\theta^+)^2(\bar\theta^+)^2D(x_A,u)\,,\end{aligned}\tag{A.2}$$

$Q^i_\alpha\Phi$ etc. are given by

$$\begin{aligned}Q^i_\alpha\Phi =\ & -u^{+i}\psi_\alpha-2u^{+i}\theta^+_\alpha M+\bar\theta^{+\dot\alpha}\left\{-u^{+i}\sigma^\mu_{\alpha\dot\alpha}A_\mu+2iu^{-i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi\right\}-\\ &-(\bar\theta^+)^2\left\{u^{+i}\lambda_\alpha+iu^{-i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\chi^{\dot\alpha}\right\}+\\ &+(\theta^+\sigma^\mu\bar\theta^+)\left\{u^{+i}(\sigma^\mu\bar\kappa)_\alpha-iu^{-i}(\partial_\nu\psi\sigma^\mu\bar\sigma^\nu)^\gamma\varepsilon_{\alpha\gamma}\right\}+\\ &+(\bar\theta^+)^2\theta^{+\beta}\left\{-u^{+i}2\varepsilon_{\alpha\beta}D-iu^{-i}(\sigma^\nu\bar\sigma^\mu)_\beta{}^\gamma\varepsilon_{\alpha\gamma}\partial_\mu A_\nu\right\}+\\ &+2iu^{-i}(\theta^+)^2\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{+\dot\alpha}\partial_\mu M-u^{-i}i(\theta^+)^2(\bar\theta^+)^2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\kappa^{\dot\alpha}\,,\end{aligned}\tag{A.3}$$

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$$
\begin{aligned}
Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}\Phi \;=\;& 2u^{+i_1}u^{+i_2}\varepsilon_{\alpha_1\alpha_2}M - \\
& -\bar\theta^{+\dot\alpha}\Big\{2u^{+i_1}u^{+i_2}\varepsilon_{\alpha_1\alpha_2}\bar\kappa_{\dot\alpha} - 2iu^{-i_1}u^{+i_2}\partial_\nu\psi_{\alpha_2}\sigma^\nu_{\alpha_1\dot\alpha} - 2iu^{+i_1}u^{-i_2}\sigma^\mu_{\alpha_2\dot\alpha}\partial_\mu\psi_{\alpha_1}\Big\} + \\
& +(\bar\theta^+)^2\Big\{2u^{+i_1}u^{+i_2}\varepsilon_{\alpha_1\alpha_2}D + iu^{-i_1}u^{+i_2}(\sigma^\nu\bar\sigma^\mu)_{\alpha_2}{}^\gamma\varepsilon_{\alpha_1\gamma}\partial_\mu A_\nu + iu^{+i_1}u^{-i_2}\times \\
& \qquad \times\varepsilon_{\alpha_1\gamma}(\sigma^\mu\bar\sigma^\nu)_{\alpha_2}{}^\gamma\partial_\mu A_\nu + 2u^{-i_1}u^{-i_2}\varepsilon_{\alpha_1\gamma}(\sigma^\mu\bar\sigma^\nu)_{\alpha_2}{}^\gamma\partial_\mu\partial_\nu\phi\Big\} + \\
& +(\theta^+\sigma^\mu\bar\theta^+)\Big\{2iu^{-i_1}u^{+i_2}\varepsilon_{\alpha_1\alpha}(\sigma^\mu\bar\sigma^\nu)_{\alpha_2}{}^\alpha\partial_\nu M - \\
& \qquad -2iu^{+i_1}u^{-i_2}\varepsilon_{\alpha_2\alpha}(\sigma^\mu\bar\sigma^\nu)_{\alpha_1}{}^\alpha\partial_\nu M\Big\} + \\
& +(\bar\theta^+)^2\theta^{+\beta}\Big\{2iu^{-i_1}u^{+i_2}\varepsilon_{\alpha_2\beta}(\sigma^\mu\partial_\mu\bar\kappa)_{\alpha_1} - 2iu^{+i_1}u^{-i_2}\varepsilon_{\alpha_1\beta}(\sigma^\mu\partial_\mu\bar\kappa)_{\alpha_2} + \\
& \qquad + 2u^{-i_1}u^{-i_2}\varepsilon_{\alpha_1\alpha_2}\partial^\mu\partial_\mu\psi_\beta\Big\} + \\
& +(\theta^+)^2(\bar\theta^+)^2\Big\{-2u^{-i_1}u^{-i_2}\varepsilon_{\alpha_1\alpha_2}\partial^\mu\partial_\mu M\Big\}, \qquad (A.4)
\end{aligned}
$$

$$
\begin{aligned}
Q^{i_3}_{\alpha_3}Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}\Phi \;=\;& -\bar\theta^{+\dot\alpha}\Big\{4iu^{-i_1}u^{+i_2}u^{+i_3}\varepsilon_{\alpha_3\alpha_2}\sigma^\mu_{\alpha_1\dot\alpha}\partial_\mu M - \\
& \quad - 4iu^{+i_1}u^{-i_2}u^{+i_3}\varepsilon_{\alpha_3\alpha_1}\sigma^\mu_{\alpha_2\dot\alpha}\partial_\mu M + \\
& \quad + 4iu^{+i_1}u^{+i_2}u^{-i_3}\varepsilon_{\alpha_1\alpha_2}\sigma^\mu_{\alpha_3\dot\alpha}\partial_\mu M\Big\} + \\
& +(\bar\theta^+)^2\Big\{-2iu^{-i_1}u^{+i_2}u^{+i_3}\varepsilon_{\alpha_2\alpha_3}(\sigma^\mu\partial_\mu\bar\kappa)_{\alpha_1} + \\
& \quad + 2iu^{+i_1}u^{-i_2}u^{+i_3}\varepsilon_{\alpha_1\alpha_3}(\sigma^\mu\partial_\mu\bar\kappa)_{\alpha_2} - \\
& \quad - 2iu^{+i_1}u^{+i_2}u^{-i_3}\varepsilon_{\alpha_1\alpha_2}(\sigma^\mu\partial_\mu\bar\kappa)_{\alpha_3} - \\
& \quad - 2u^{-i_1}u^{-i_2}u^{+i_3}\varepsilon_{\alpha_1\alpha_2}\partial^\mu\partial_\mu\psi_{\alpha_3} + \\
& \quad + 2u^{-i_1}u^{+i_2}u^{-i_3}\varepsilon_{\alpha_1\alpha_3}\partial^\mu\partial_\mu\psi_{\alpha_2} + \\
& \quad + 2u^{+i_1}u^{-i_2}u^{-i_3}\varepsilon_{\alpha_2\alpha_3}\partial^\mu\partial_\mu\psi_{\alpha_2}\Big\} + \\
& +(\bar\theta^+)^2\theta^{+\beta}\Big\{4u^{-i_1}u^{-i_2}u^{+i_3}\varepsilon_{\alpha_3\beta}\varepsilon_{\alpha_1\alpha_2}\partial^\mu\partial_\mu M - \\
& \quad - 4u^{-i_1}u^{+i_2}u^{-i_3}\varepsilon_{\beta\alpha_2}\varepsilon_{\alpha_1\alpha_3}\partial^\mu\partial_\mu M + \\
& \quad + 4u^{+i_1}u^{-i_2}u^{-i_3}\varepsilon_{\beta\alpha_1}\varepsilon_{\alpha_2\alpha_3}\partial^\mu\partial_\mu M\Big\}, \qquad (A.5)
\end{aligned}
$$

$$
\begin{aligned}
Q^{i_4}_{\alpha_4}Q^{i_3}_{\alpha_3}Q^{i_2}_{\alpha_2}Q^{i_1}_{\alpha_1}\Phi \;=\;& (\bar\theta^+)^2\Big\{-4u^{-i_1}u^{-i_2}u^{+i_3}u^{+i_4}\varepsilon_{\alpha_3\alpha_4}\varepsilon_{\alpha_1\alpha_2} + \\
& \quad + 4u^{-i_1}u^{+i_2}u^{-i_3}u^{+i_4}\varepsilon_{\alpha_4\alpha_2}\varepsilon_{\alpha_1\alpha_3} - \\
& \quad - 4u^{+i_1}u^{-i_2}u^{-i_3}u^{+i_4}\varepsilon_{\alpha_4\alpha_1}\varepsilon_{\alpha_2\alpha_3} - \\
& \quad - 4u^{+i_1}u^{+i_2}u^{+i_3}u^{-i_4}\varepsilon_{\alpha_3\alpha_2}\varepsilon_{\alpha_1\alpha_4} + \\
& \quad + 4u^{+i_1}u^{-i_2}u^{+i_3}u^{-i_4}\varepsilon_{\alpha_3\alpha_1}\varepsilon_{\alpha_2\alpha_4} - \\
& \quad - 4u^{+i_1}u^{+i_2}u^{-i_3}u^{-i_4}\varepsilon_{\alpha_1\alpha_2}\varepsilon_{\alpha_3\alpha_4}\Big\}\partial^\mu\partial_\mu M\,. \qquad (A.6)
\end{aligned}
$$

B. Calculation of third order lagrangian

In this appendix we calculate the action which is third order of V^{++}_{WZ}:

$$
S_{*,3} = \frac{i}{6}\int d^4x d^8\theta du_1 du_2 du_3 \frac{V^{++}_{WZ}(1) * V^{++}_{WZ}(2) * V^{++}_{WZ}(3)}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}\,. \qquad (B.1)
$$

We use the chiral basis to compute this action. In the chiral basis, the gauge superfield V_{WZ}^{++} is expressed as

$$
\begin{aligned}
V_{WZ}^{++}(x_L^\mu, u) = & -2i\theta^+\sigma^\mu\bar{\theta}^+ A_\mu - 2(\bar{\theta}^+)^2\theta^+\sigma^\mu\bar{\sigma}^\nu\theta^-\partial_\nu A_\mu + i\sqrt{2}(\bar{\theta}^+)^2\phi - \\
& - i\sqrt{2}(\theta^+)^2\bar{\phi} - 2\sqrt{2}(\theta^+)^2\theta^-\sigma^\mu\bar{\theta}^+\partial_\mu\bar{\phi} - \sqrt{2}i(\theta^+)^2(\theta^-)^2(\bar{\theta}^+)^2\partial^2\bar{\phi} + \\
& + 4(\bar{\theta}^+)^2\theta^+\psi^- - 4(\theta^+)^2\bar{\theta}^+\bar{\psi}^- - 4i(\theta^+)^2(\bar{\theta}^+)^2\theta^-\sigma^\mu\partial_\mu\bar{\psi}^- + \\
& + 3(\theta^+)^2(\bar{\theta}^+)^2 D^{--}\,. \qquad \text{(B.2)}
\end{aligned}
$$

We investigate what kind of contributions from $(V_{WZ}^{++})_*^3$ exist in the lagrangian. As seen from the arguments in section 3, we need to compute the contributions from P and P^3 in the $*$-product.

Firstly we study the contributions from single P. The $*$-product formulas for first order of P is summarized as follows:

$$
\begin{aligned}
\theta_1^{+\alpha}\ P\ \theta_2^{+\beta} &= +\frac{1}{2}C_{12}^{++\alpha\beta}\,, && \text{(B.3)}\\
\theta_1^{+\beta}\theta_{1\ \gamma}^{-}\ P\ \theta_2^{+\alpha} &= -\frac{1}{2}C_{12}^{++\beta\alpha}\theta_{1\ \gamma}^{-} + \frac{1}{2}C_{12}^{-+\alpha'\alpha}\varepsilon_{\gamma\alpha'}\theta_1^{+\beta}\,, && \text{(B.4)}\\
\theta_1^{+\alpha}\ P\ (\theta_2^+)^2 &= +C_{12}^{++\alpha\alpha'}\theta_{2\ \alpha'}^{+}\,, && \text{(B.5)}\\
(\theta_1^+)^2\ P\ \theta_2^{+\alpha} &= -C_{12}^{++\alpha'\alpha}\theta_{1\ \alpha'}^{+}\,, && \text{(B.6)}\\
\theta_1^{+\alpha}\theta_{1\ \beta}^{-}\ P\ \theta_2^{+\gamma}\theta_{2\ \delta}^{-} &= -\frac{1}{2}C_{12}^{++\alpha\gamma}\theta_{1\ \beta}^{-}\theta_{2\ \delta}^{-} + \frac{1}{2}C_{12}^{+-\alpha\alpha'}\varepsilon_{\delta\alpha'}\theta_{1\ \beta}^{-}\theta_2^{+\gamma} + \\
&\quad + \frac{1}{2}C_{12}^{-+\alpha'\gamma}\varepsilon_{\beta\alpha'}\theta_1^{+\alpha}\theta_{2\ \delta}^{-} - \frac{1}{2}C_{12}^{--\alpha'\beta'}\varepsilon_{\beta\alpha'}\varepsilon_{\delta\beta'}\theta_1^{+\alpha}\theta_2^{+\gamma}\,, && \text{(B.7)}\\
\theta_1^{+\alpha}\theta_{1\ \beta}^{-}\ P\ (\theta_2^+)^2 &= -C_{12}^{++\alpha\alpha'}\theta_{1\ \beta}^{-}\theta_{2\ \alpha'}^{+} + C_{12}^{-+\alpha'\beta'}\varepsilon_{\beta\alpha'}\theta_1^{+\alpha}\theta_{2\ \beta'}^{+}\,, && \text{(B.8)}\\
(\theta_1^+)^2\ P\ (\theta_2^+)^2 &= -2C_{12}^{++\alpha'\beta'}\theta_{1\ \alpha'}^{+}\theta_{2\ \beta'}^{+}\,, && \text{(B.9)}\\
(\theta_1^+)^2(\theta_1^-)^2\ P\ (\theta_2^+)^2 &= -2C_{12}^{++\alpha'\beta'}\theta_{1\ \alpha'}^{+}(\theta_1^-)^2\theta_{2\ \beta'}^{+} - 2C_{12}^{-+\alpha'\beta'}(\theta_1^+)^2\theta_{1\ \alpha'}^{-}\theta_{2\ \beta'}^{+}\,, && \text{(B.10)}\\
(\theta_1^+)^2\theta_1^{-\alpha}\ P\ (\theta_2^+)^2\theta_2^{-\beta} &= +2C_{12}^{++\alpha'\beta'}\theta_{1\ \alpha'}^{+}\theta_1^{-\alpha}\theta_{2\ \beta'}^{+}\theta_2^{-\beta} + C_{12}^{+-\alpha'\beta}\theta_{1\ \alpha'}^{+}\theta_1^{-\alpha}(\theta_2^+)^2 + \\
&\quad + C_{12}^{-+\alpha\alpha'}(\theta_1^+)^2\theta_{2\ \alpha'}^{+}\theta_2^{-\beta} + \frac{1}{2}C_{12}^{--\alpha\beta}(\theta_1^+)^2(\theta_2^+)^2\,. && \text{(B.11)}
\end{aligned}
$$

The following identities are also useful:

$$
\begin{aligned}
(\theta_1^+)^2(\theta_2^\pm)^2 &= (u_1^+u_2^\pm)^2\theta^4\,, && \text{(B.12)}\\
(\theta_1^+)^2\theta_2^{\pm\alpha}\theta_3^{+\beta} &= +\frac{1}{2}\varepsilon^{\alpha\beta}\theta^4(u_1^+u_2^\pm)(u_3^+u_1^+)\,, && \text{(B.13)}\\
(\bar{\theta}_1^+)^2\bar{\theta}_2^{\pm\dot{\alpha}}\bar{\theta}_3^{+\dot{\beta}} &= -\frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^4(u_1^+u_2^\pm)(u_3^+u_1^+)\,. && \text{(B.14)}
\end{aligned}
$$

We also use some the harmonic integral formulas, summarized as

$$
\begin{aligned}
\int du\, u_i^+u_j^- &= \frac{1}{2!}\epsilon_{ij}\,, && \text{(B.15)}\\
\int du\, u_i^+u_j^+u_k^-u_l^- &= \frac{1}{3!}(\epsilon_{ik}\epsilon_{jl} + \epsilon_{il}\epsilon_{jk})\,, && \text{(B.16)}\\
\int du_1\,\frac{u_1^+}{u_1^+u_2^+} &= -u_2^-\,, && \text{(B.17)}
\end{aligned}
$$

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$$\int du_3 \, \frac{u_3^{+(i}u_3^{+j)}}{(u_3^+u_1^+)(u_2^+u_3^+)} = \frac{u_1^{+(i}u_1^{-j)} - u_2^{+(i}u_2^{-j)}}{u_1^+u_2^+}\,, \tag{B.18}$$

From the single P, we have thirteen types contributions to the lagrangian, listed as below:

$$\begin{aligned}
L_1 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times 4\sqrt{2}i\left\{\theta_1^+\sigma^\mu\bar{\theta}_1^+ \; P \; \theta_2^+\sigma^\nu\bar{\theta}_2^+\right\}(\theta_3^+)^2(\theta_3^-)^2(\bar{\theta}_3^+)^2 A_\mu A_\nu \,\partial^2\bar{\phi}\,, && \text{(B.19)}\\
L_2 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times 32\left\{\theta_1^+\sigma^\mu\bar{\sigma}^\nu\theta_1^- \; P \; (\theta_2^+)^2\bar{\theta}^+_{2\,\dot{\alpha}}\right\}(\theta_3^+)^2\bar{\theta}^+_{3\,\dot{\beta}}\partial_\nu A_\mu \bar{\psi}_2^{-\,\dot{\alpha}}\bar{\psi}_3^{-\,\dot{\beta}}\,, && \text{(B.20)}\\
L_3 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times(-48)\left\{(\theta_1^+)^2\bar{\theta}^+_{1\,\dot{\alpha}} \; P \; (\theta_2^+)^2\bar{\theta}^+_{2\,\dot{\beta}}\right\}(\theta_3^+)^2(\bar{\theta}_3^+)^2\bar{\psi}_1^{-\,\dot{\alpha}}\bar{\psi}_2^{-\,\dot{\beta}}D_{33}^{--}\,, && \text{(B.21)}\\
L_4 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times(-4\sqrt{2}i)\left\{(\bar{\theta}_1^+)^2\theta_1^+\sigma^\mu\bar{\sigma}^\nu\theta_1^- \; P \; \theta_2^+\sigma^\rho\bar{\sigma}^\sigma\theta_2^-(\bar{\theta}_2^+)^2\right\}(\theta_3^+)^2\partial_\nu A_\mu\partial_\sigma A_\rho\,\bar{\phi}\,, && \text{(B.22)}\\
L_5 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times(-9\sqrt{2}i)\left\{(\theta_1^+)^2(\bar{\theta}_1^+)^2 \; P \; (\theta_2^+)^2\right\}(\theta_3^+)^2(\bar{\theta}_3^+)^2 D_{11}^{--}\bar{\phi}D_{33}^{--}\,, && \text{(B.23)}\\
L_6 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times(-32)\left\{\theta_1^+\sigma^\mu\bar{\theta}_1^+ \; P \; (\theta_2^+)^2\bar{\theta}^+_{2\,\dot{\alpha}}\right\}(\theta_3^+)^2(\bar{\theta}_3^+)^2(\theta_3^-\sigma^\nu)_{\dot{\beta}}A_\mu\bar{\psi}_2^{-\,\dot{\alpha}}\partial_\nu\bar{\psi}_3^{-\,\dot{\beta}}\,, && \text{(B.24)}\\
L_7 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times 16\sqrt{2}\left\{(\bar{\theta}_1^+)^2\theta_1^{+\alpha} \; P \; (\theta_2^+)^2\right\}(\theta_3^+)^2(\theta_3^-\sigma^\mu)_{\dot{\alpha}}(\bar{\theta}_3^+)^2\psi^-_{1\,\alpha}\bar{\phi}\,\partial_\mu\bar{\psi}_3^{-\,\dot{\alpha}}\,, && \text{(B.25)}\\
L_8 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times(-32)\left\{\sqrt{2}i(\bar{\theta}_1^+)^2\theta_1^{+\alpha} \; P \; (\theta_2^+)^2\bar{\theta}^+_{2\,\dot{\alpha}}\right\}(\theta_3^+)^2\theta_3^-\sigma^\nu\bar{\theta}_3^+\psi^-_{1\,\alpha}\bar{\psi}_2^{-\,\dot{\alpha}}\,\partial_\mu\bar{\phi}\,, && \text{(B.26)}\\
L_9 &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times(-8\sqrt{2}i)\left\{(\bar{\theta}_1^+)^2\theta_1^+\sigma^\mu\bar{\sigma}^\nu\theta_1^- \; P \; \theta_2^+\sigma^\rho\bar{\theta}_2^+\right\}(\theta_3^+)^2\theta_3^-\sigma^\sigma\bar{\theta}_3^+\partial_\nu A_\mu A_\rho\partial_\sigma\bar{\phi}\,, && \text{(B.27)}\\
L_{10} &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \\
&\times 12\sqrt{2}i\left\{(\theta_1^+)^2(\bar{\theta}_1^+)^2 \; P \; \theta_2^+\sigma^\mu\bar{\theta}_2^+\right\}(\theta_3^+)^2\theta_3^-\sigma^\nu\bar{\theta}_3^+D_{11}^{--}A_\mu\partial_\nu\bar{\phi}\,, && \text{(B.28)}\\
L_{11} &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \\
&\times 6\sqrt{2}i\left\{(\bar{\theta}_1^+)^2\theta_1^+\sigma^\mu\bar{\sigma}^\nu\theta_1^- \; P \; (\theta_2^+)^2\right\}(\theta_3^+)^2(\bar{\theta}_3^+)^2\partial_\nu A_\mu\bar{\phi}D_{33}^{--}\,, && \text{(B.29)}\\
L_{12} &= \frac{i}{6}\int d^8\theta \frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times
\end{aligned}$$

$$
\begin{aligned}
&\times 8\sqrt{2}i\left\{(\theta_1^+)^2\theta_1^-\sigma^\mu\bar\theta_1^+\ P\ (\theta_2^+)^2\theta_2^-\sigma^\nu\bar\theta_2^+\right\}(\bar\theta_3^+)^2\partial_\mu\bar\phi\,\partial_\nu\bar\phi\,\phi\,, && \text{(B.30)}\\
L_{13} &= \frac{i}{6}\int d^8\theta\frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}\times\\
&\times(-2\sqrt{2}i)\left\{(\theta_1^+)^2(\theta_1^-)^2(\bar\theta_1^+)^2\ P\ (\theta_2^+)^2\right\}(\bar\theta_3^+)^2\partial^2\bar\phi\,\bar\phi\,\phi\,. && \text{(B.31)}
\end{aligned}
$$

These terms are computed by applying the formulas (B.3)–(B.18). The results are as follows:

$$
\begin{aligned}
L_1 &= -\frac{\sqrt{2}}{12}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}C^{\alpha\beta}_{ij}\epsilon^{ij}A_\mu A_\nu\partial^2\bar\phi\,,\\
L_2 &= -\frac{2i}{3}\,C^{\mu\nu}_{(ij)}\partial_\mu A_\nu\bar\psi^i\bar\psi^j\,,\\
L_3 &= \frac{i}{6}C_s\bar\psi^i\bar\psi^j D_{ij}\,,\\
L_5 &= -\frac{\sqrt{2}}{12}C_s\,D_{ij}\,D^{ij}\,\bar\phi\,,\\
L_6 &= -\frac{i}{12}C_s\varepsilon^{\alpha\beta}A_\mu(\sigma^\mu\bar\psi^i)_\alpha(\sigma^\nu\partial_\nu\bar\psi_i)_\beta+\frac{i}{9}\,C^{(\alpha\beta)}_{(ij)}A_\mu(\sigma^\mu\bar\psi^i)_\alpha(\sigma^\nu\partial_\nu\bar\psi^j)_\beta\,,\\
L_7 &= \frac{\sqrt{2}}{12}i\,C_s\,\psi^i\sigma^\mu\partial_\mu\bar\psi_i\,\bar\phi-\frac{\sqrt{2}}{9}i\,C^{(\alpha\beta)}_{(ij)}\,\bar\phi\,\psi^i_\alpha(\sigma^\mu\partial_\mu\bar\psi^j)_\beta\,,\\
L_8 &= \frac{\sqrt{2}}{12}i\,C_s\psi^i\sigma^\mu\bar\psi_i\,\partial_\mu\bar\phi-\frac{\sqrt{2}i}{3}\,C^{\alpha\beta}_{(ij)}\psi^i_\alpha(\sigma^\mu\bar\psi^j)_\beta\,\partial_\mu\bar\phi\,,\\
L_9 &= -\frac{\sqrt{2}}{6}\,C_s\mathrm{Tr}\,(\sigma^{\mu\nu}\sigma^{\rho\sigma})\partial_\mu A_\nu A_\sigma\partial_\rho\bar\phi\,,\\
L_{10} &= -\frac{\sqrt{2}}{6}\,C^{\mu\nu}_{(ij)}D^{ij}A_\nu\partial_\mu\bar\phi\,,\\
L_{11} &= \frac{\sqrt{2}}{6}\,C^{\mu\nu}_{(ij)}\partial_\mu A_\nu\,\bar\phi\,D^{ij}\,,\\
L_4 &= L_{12}=L_{13}=0\,. \qquad\qquad \text{(B.32)}
\end{aligned}
$$

The contributions to the lagrangian coming from P^3 are two types:

$$
\begin{aligned}
L_{14} &= \frac{i}{6}\int d^8\theta\frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}(-8\sqrt{2}i)\times\\
&\times\left\{(\theta_1^+)^2\theta_1^-\sigma^\mu\bar\theta_1^+\ \frac{1}{3!}P^3\ (\theta_2^+)^2\theta_2^-\sigma^\nu\bar\theta_2^+)^2\right\}(\theta_3^+)^2(\theta_3^-)^2(\bar\theta_3^+)^2\,\partial_\mu\bar\phi\,\partial_\nu\bar\phi\,\partial^2\bar\phi\,, && \text{(B.33)}\\
L_{15} &= \frac{i}{6}\int d^8\theta\frac{du_1du_2du_3}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}\times\\
&\times 2\sqrt{2}i\left\{(\theta_1^+)^2(\theta_1^-)^2(\bar\theta_1^+)^2\ \frac{1}{3!}P^3\ (\theta_2^+)^2(\theta_2^-)^2(\bar\theta_2^+)^2\right\}(\theta_3^+)^2\partial^2\bar\phi\,\partial^2\bar\phi\,\bar\phi\,. && \text{(B.34)}
\end{aligned}
$$

Using the formulas

$$
\begin{aligned}
(\theta_1^+)^2\theta_1^{-\,\alpha}\ P^3\ (\theta_2^+)^2\theta_2^{-\,\beta} &= -3\det C^{++}_{12}C^{--\,\alpha\beta}_{12}+\\
&\quad +3C^{++\gamma'\delta'}_{12}C^{+-\,\beta'\beta}_{12}C^{-+\alpha\alpha'}_{12}\varepsilon_{\gamma'\beta'}\varepsilon_{\delta'\alpha'}\,,\\
(\theta_1^+)^2(\theta_1^-)^2P^3(\theta_2^+)^2(\theta_2^-)^2 &= -6C^{\alpha\delta}_{12}C^{\beta\epsilon}_{21}C^{\gamma\zeta}_{22}\varepsilon_{\alpha\epsilon}\varepsilon_{\delta\zeta}\varepsilon_{\beta\gamma}(\theta^1)^2+
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad +6C^{\alpha\delta}_{11} C^{\beta\epsilon}_{12} C^{\gamma\zeta}_{21}\, \varepsilon_{\alpha\beta}\varepsilon_{\delta\zeta}\varepsilon_{\epsilon\gamma}(\theta^2)^2\,,\\
\int du_1\, \frac{u_1^{+i}u_1^{+k}u_1^{-m}}{u_1^+u_2^+} &= -\frac{1}{2}\left(u_2^{+(i}u_2^{-k}u_2^{-m)} + \frac{4}{3}\epsilon^{m(k}u_2^{-i)}\right),\\
\int du_1 du_2\, \frac{u_1^+{}_i u_1^+{}_k u_1^{-m} u_2^{+j} u_2^{+l} u_2^{-n}}{u_1^+u_2^+} &= +\frac{1}{24}\epsilon^{m(l}\delta_k{}^j\delta_i{}^{n)} + \frac{2}{9}\epsilon^{n(j}\delta_{(i}{}^{l)}\delta_{k)}{}^m\,.
\end{aligned}
\tag{B.35}
$$

we find $L_{14} = L_{15} = 0$. Including the combinatorial factor, the third order lagrangian $\mathcal{L}_3$ becomes

$$
\begin{aligned}
\mathcal{L}_3 &= 3\times\sum_{i=1}^{6} L_i + 6\times\sum_{i=7}^{15} L_i\\
&= +\frac{\sqrt{2}}{4}C_s A_\mu A^\mu \partial^2\bar\phi - \frac{i}{2}C_s\varepsilon^{\alpha\beta}A_\mu(\sigma^\mu\bar\psi^i)_\alpha(\sigma^\nu\partial_\nu\bar\psi_i)_\beta + \frac{2}{3}i\,C^{(\alpha\beta)}_{(ij)}A_\mu(\sigma^\mu\bar\psi^i)_\alpha(\sigma^\nu\partial_\nu\bar\psi^j)_\beta -\\
&\quad -2iC^{\mu\nu}_{(ij)}\partial_\mu A_\nu\bar\psi^i\bar\psi^j + \frac{i}{2}C_s\bar\psi^i\bar\psi^j D_{ij} + \frac{\sqrt{2}}{2}i\,C_s\psi^i\sigma^\mu\partial_\mu\bar\psi_i\bar\phi - \frac{2\sqrt{2}}{3}i\,C^{\alpha\beta}_{(ij)}\psi^i_\alpha(\sigma^\mu\partial_\mu\bar\psi^j)_\beta\bar\phi +\\
&\quad +\frac{\sqrt{2}}{2}i\,C_s\psi^i\sigma^\mu\bar\psi_i\partial_\mu\bar\phi - 2\sqrt{2}i\,C^{\alpha\beta}_{(ij)}\psi^i_\alpha(\sigma^\mu\bar\psi^j)_\beta\partial_\mu\bar\phi - \sqrt{2}\,C_s\mathrm{Tr}\,(\sigma^{\mu\nu}\sigma^{\rho\sigma})\partial_\mu A_\nu A_\sigma\partial_\rho\bar\phi -\\
&\quad -\sqrt{2}\,C^{\mu\nu}_{(ij)}D^{ij}A_\nu\partial_\mu\bar\phi + \sqrt{2}\,C^{\mu\nu}_{(ij)}\partial_\mu A_\nu\bar\phi D^{ij} - \frac{\sqrt{2}}{4}C_s D_{ij}D^{ij}\bar\phi\,.
\end{aligned}
\tag{B.36}
$$

This is shown to be equal to (3.25) with the use of the formula $\mathrm{Tr}\,(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}$.

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